Part III: Analysis of Partial Differential Equations

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Contents

LECTURE 1

1 Introduction

Organization & Resources

Comments and corrections regarding these lecture notes are very welcome and may be sent to ga482@cam.ac.uk.

Example classes. Four example classes, each lasting two hours, will be held during the term. For each class I will upload in advance on Moodle a corresponding Problem Sheet with exercises for you to solve and we will discuss them in the next example class. Furthermore, you can hand in the exercises denoted with ($\star$) to me for feedback.

Resources. Lecture notes will be made available on Moodle after each class. Recordings of the lectures can also be found there; nevertheless, regular attendance is strongly recommended, as it substantially improves comprehension and long-term retention. The course has previously been taught by Clément Mouhot, Claude Warnick, and Zoe Wyatt, whose lecture notes are available on their respective webpages.

These lecture notes make no claim of originality. They draw largely on past lecture notes, particularly by C. Mouhot, for the same course.

The principal references are:

- **L. C. Evans**, *Partial Differential Equations.* Perhaps the most widely used reference and the closest in spirit to this course.
- **H. Brezis**, *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* A standard reference, particularly in France, emphasizing the connection between Functional Analysis and PDEs, and developing Sobolev spaces from one to several dimensions.
- **F. John**, *Partial Differential Equations.* A classical source, especially for the Cauchy-Kovalevskaya theorem and the method of characteristics.
- **E. Lieb and M. Loss**, *Analysis.*
- **S. Klainerman**, *Partial Differential Equations (Princeton Companion to Mathematics).* An introductory essay offering a broad overview of the field of partial differential equations.

1.1 Overview of the course

Historical remarks. From infinitesimal calculus to PDEs The modern *Analysis of Partial Differential Equations* (PDEs) originates in the late 17th century with Newton and Leibniz, whose invention of differential calculus (building on earlier ideas such as Archimedes' method of exhaustion) marked the birth of modern physics. Calculus provided a universal language to describe continuous change, allowing physical laws to be formulated as differential equations (initially in a single variable). These equations could then be studied to make both qualitative and quantitative predictions, laying the foundation of modern physics.

In the early 18th century, Euler, N. Bernoulli and d'Alembert, analyzed equations involving differentiation with respect to several variables, motivated by fundamental physical problems such as vibrating strings and fluid motion.

The 19th century introduced rigorous analysis (e.g. with the works of Cauchy, Fourier and G. Green) and 20th century saw the development of the frameworks of functional analysis and distributions (e.g. with the works of Hilbert, Sobolev, Schwartz), which enabled the modern study of well-posedness and regularity of solutions.[1]

Motivations for PDEs Many mathematical fields are fully connected with the study of PDEs through the study of physical laws, as Fourier analysis (born to study the conduction of heat), complex analysis (which depends on the Cauchy-Riemann equations), and functional analysis (serving as setting for the modern approach to PDE, but also important in quantum physics) and differential geometry (e.g. studying the Ricci flow as PDE has been fundamental to solve the Poincaré conjecture).

The fundamental laws of physics typically express relations among quantities depending on multiple independent variables and their partial derivatives. This naturally gives rise to PDEs such as the Laplace, Euler, Navier–Stokes, Maxwell, Boltzmann, Einstein and Schrödinger.

The analysis of PDEs thus forms a vast and central discipline, standing at the crossroads of physics and numerous branches of pure and applied mathematics.

Modern approach vs classical approach In this introductory course, we will focus on the *modern approach* to PDE theory. This contrasts with the *classical approach*, which seeks explicit solution formulas using methods such as Fourier series, integral transforms, and separation of variables (as you may have encountered in undergraduate courses). While effective in specific cases, these methods are not sufficient to understand PDEs within a broader framework.

The modern approach instead studies existence, uniqueness, and qualitative properties of solutions by embedding the problem into a "suitable" function space equipped with the right topology. Choosing this "suitable" space (and associated topology) is a crucial part of the problem: it must be large enough to contain some solutions, yet restrictive enough to guarantee uniqueness. Moreover, this approach has the additional advantage of extending the notion of solution to functions less regular than classical differentiability would require.

Content of the course After a first introduction to PDEs starting from your knowledge of ODEs, we will first present the only general result available for PDEs (in a very special class of solutions), then, after introducing the necessary functional tools, we will focus on linear elliptic and evolutionary problems through energy estimates. We will focus on Laplace equation, wave equation and Burgers equation. In the example classes, we will touch also linear parabolic problems and few nonlinear scenarios.

- *Chapter 1: Introduction (2 lectures)* Starting from your knowledge of ordinary differential equations (ODEs) we will introduce the concept of PDEs. We also present some fundamental examples.
- *Chapter 2: The Cauchy-Kovalevskaya theory (4 lectures)* We present basically the only general existence theorem for PDEs. This is about solving locally PDEs with analytic coefficient within the analytic class of solutions. It was first proved by Cauchy for a special class of PDEs, and later generalized by Kovalevskaya to its modern form. In her work, she also clarified the geometric condition required for the theorem to hold.
- *Chapter 3: Functional toolbox (5 lectures)* This chapter prepares for the following two. It reviews key definitions and properties of Hölder and Lebesgue spaces, introduces

[1]For more historical information on the period before Euler see Cajori, Amer. Math. Monthly 35 (1928), instead on the period after Euler, see Brezis-Browder, Adv. Math. 135 (1998).

weak (generalised) derivatives, and studies them in three settings: (1) approximation by convolution, (2) extension and trace theorems, and (3) Sobolev spaces and their inequalities.

- *Chapter 4: Elliptic PDEs (6 lectures)* We study variants of the Laplace equation with prescribed boundary conditions. We introduce the Lax-Milgram theorem to construct solution and its extensions through the Fredholm theory. We also discuss the elliptic regularity estimates.
- *Chapter 5: Hyperbolic PDEs (7 lectures)* We study two fundamental hyperbolic equations, the transport equation (first order) and the wave equation (second order). We also discuss the method of characteristic and formation of singularity for Burgers equation. We also present the method of energy estimate for the wave equation and its consequences.

1.2 From ODEs to PDEs

Consider a function

$$F(x, y_1, \dots, y_{k+1}),$$

depending on $k+2$ real variables. An ordinary differential equation (ODE) is an equation of the form

$$F\big(x, u(x), u'(x), \dots, u^{(k)}(x)\big) = 0, \quad x \in \mathcal{U} \subset \mathbb{R},$$

for some k-times differentiable $u : \mathcal{U} \to \mathbb{R}$ (classical solution), where $\mathcal{U}$ is an interval here.

Example 1.1. if $F(x, y, z) = f(x, y) - z$, we recover

$$y'(x) = f(x, y(x)),$$

whose solutions $(x, y(x))$ are trajectories (integral curves).

Partial differential equations (PDEs) arise when the unknown depends on several real variables.

$$u = u(x) = u(x_1, \dots, x_n), \quad n \geq 2,$$

The equation then involves the partial derivatives

$$\frac{\partial u}{\partial x_i}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_\ell}, \dots$$

In this case we work with a domain (open, not empty connected set) $\mathcal{U} \subset \mathbb{R}^n$.

Definition 1.2 (Order-k PDE)**.** Let $n \geq 2$ and $\mathcal{U} \subset \mathbb{R}^n$ be a domain. A **partial differential equation of order/rank** k is a relation

$$F\big(x, u(x), \nabla u(x), \dots, \nabla^k u(x)\big) = 0, \qquad x \in \mathcal{U}, \tag{1.1}$$

where $F : \mathcal{U} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$ and the unknown is $u : \mathcal{U} \to \mathbb{R}$. A *classical solution* is a function $u \in C^k(\mathcal{U})$ for which (1.1) holds pointwise on $\mathcal{U}$ after substituting $u, \nabla u, \dots, \nabla^k u$.

Remark 1.3. 1. (Notation) $\nabla u = (\partial_{x_i} u)_i$, $\nabla^2 u = (\partial_{x_i x_j} u)_{ij}$, and in general $\nabla^k u$ collects all order-k partial derivatives.

2. (Evolution form) If one coordinate can be singled out as time, say $t = x_1$, the PDE may be written as an *evolution equation* (e.g. $\partial_t u = G(t, x', u, \nabla_{x'} u, \dots, D^k_{x'} u)$ where $x' = (x_2, \dots, x_n)$). Finding a variable that plays the role of time can be a difficult task, as for the Einstein equation in general relativity.
3. (Data) For ODEs, a solution depends by an initial value, $u(t_0) = u_0$. For PDEs the picture is richer. If the equation is already in evolution form, this often reduces to prescribing the *initial profile* $u(t_0, \cdot)$ over the $n-1$ remaining variables (and, when needed, time-derivative data), while *boundaries may occur in other variables* (e.g. on a time–space cylinder) and require boundary conditions there. In full generality, the data must be given on a *noncharacteristic hypersurface*, a geometric non-degeneracy (detailed in the next chapter).
4. (Systems) More generally, $u : \mathcal{U} \to \mathbb{R}^m$ and F may be $\mathbb{R}^N$-valued, yielding a system of PDEs.
5. (Infinite-dimensional dynamical system viewpoint) It is natural to ask whether a PDE can be seen as an infinite-dimensional version of an ODE. In the simplest case, whenever a PDE can be written in evolution form, e.g.

$$\partial_t u = G(x', u, D_{x'} u, \dots, D^{(k)}_{x'} u),$$

we may regard $u(t)$ as a curve in a function space X (equipped with a suitable topology), and $G : X \to X$ as an operator, possibly nonlinear. However, even in this case, it is too naive to think of PDEs as just an infinite dimensional version of ODEs. PDEs bring in additional difficulties and features.

(a) *Geometry via the principal symbol:* The loss of total order (when passing from $\mathbb{R}$ to higher dimension) gives new geometric phenomena (elliptic/parabolic/hyperbolic types) which imply relevant physical features (such as time reversibility or irreversibility, finite or infinite speed of propagation).

(b) *Functional setting.* In infinite dimensions, norms are no longer equivalent. Even linear operators such as derivatives act as unbounded operators, so the choice of function spaces and topologies for their domains and ranges becomes a crucial part of the analysis.

(c) *Nonlinearity:* interactions between derivatives creates much more variety than just sign and modulus of the nonlinearity (as in the ODEs case).[2]
6. (Unifying principles in PDE) There is no single, systematic theory as for ODEs beyond a few foundational results. One must exploit the equation's structure (elliptic/parabolic/hyperbolic) and its properties like scaling, invariances, type of data and geometry, hence focusing on *fundamental equations* and faithful toy problems, rather than randomly invented ones. The field is unified by goals and methods, not by an universal theory.

[2]At linear level the spectral theory can be seen as a generalization of the finite dimensional case, but at a very limited extent (indeed, for example, the spectrum of an operator in infinite dimension can be continuous and eigenfunctions may fail to exist).

LECTURE 2

1.3 The Cauchy problem

1.3.1 What can we learn from ODEs?

A basic question in mathematical analysis how t "invert" differentiation. Given a continuous function $F : \mathbb{R} \to \mathbb{R}$, one seeks a differentiable function $u : \mathbb{R} \to \mathbb{R}$ such that

$$u'(t) = F(t), \qquad t \in I,$$

on some open interval $I \subset \mathbb{R}$. This is the simplest type of differential equation, and the *fundamental theorem of calculus* gives the explicit solution

$$u(t) = \int_{t_0}^{t} F(y)\,\mathrm{d}y + u(t_0), \qquad t_0 \in \mathbb{R}.$$

This is a one-parameter family, determined by the **boundary condition** $u(t_0) = u_0$ (which, in this case we usually call initial condition). The argument directly yields existence and uniqueness of a differentiable solution u satisfying $u(t_0) = u_0$.

The same reasoning applies even if F is only (Riemann or Lebesgue) *integrable*. In both integration theories, the integral is obtained as a limit process (in the Riemann sense as the limit of Riemann sums and in the Lebesgue sense as the limit of integrals of simple functions), so the construction of u is intrinsically based on a limiting process.

The actual theory of ODEs begins when the right-hand side depends *locally* on the unknown function, i.e.

$$u'(t) = F(t, u(t)),$$

with F defined, for instance, on $\mathbb{R} \times \mathbb{R}$. We are interested in the existence, uniqueness, and continuation of solutions, depending on the regularity of F. The following are three classical theorems in ODEs, presented in decreasing order of regularity on F.[3]

Theorem 1.4 (Cauchy-Kovalevskaya theorem for ODEs[4]). *Assume that the vector field $F(t, u)$ is* real analytic[5] *in a neighbourhood of (t_0, u_0). Then there exists a unique local real analytic solution u of*

$$u'(t) = F(t, u(t)), \qquad u(t_0) = u_0,$$

defined in a neighbourhood of t_0.

This theorem represents the first attempt to solve nonlinear ODEs "semi-explicitly" (explicit up to summing an infinite convergent series) by infinite Taylor expansions.[6] Although theoretically important, it is not practical (in particular u is constructed by power series involving all derivatives of F) and requires very strong assumptions on F. The same analytic approach will reappear for certain PDEs in the next chapter.

[3]Another more important and more recent result, the DiPerna-Lions theorem (1989), gives existence and uniqueness of solutions in an appropriate *almost everywhere sense* to ODE when, for instance, $F = F(u) \in W^{1,1}$ and $\nabla \cdot F = 0$

[4]First proved by Cauchy (1842) for ODEs and first-order quasilinear PDEs, and extended to its modern form by Kovalevskaya (1875).

[5]A real function is *analytic* at a point if it has derivatives of all orders and coincides with its Taylor series in a neighbourhood of that point.

[6]Historically, after finding examples of ODEs (e.g. some Riccati equations, Airy equations) whose solutions could not be express in terms of elementary functions and their integrals, practice shifted from closed forms to special function and series solutions.

Theorem 1.5 (Cauchy–Lipschitz / Picard–Lindelöf theorem[7]). *Assume that the vector field* $F = F(t,u)$ *is continuous in both variables and* Lipschitz continuous *in* u. *Then for every* (t_0, u_0) *there exists a neighbourhood of* t_0 *where the ODE admits a unique* C^1 *solution* u *satisfying* $u(t_0) = u_0$. *Moreover, the solution depends continuously on the initial data* u_0.

This result, proved by a contraction mapping argument, is the most useful in applications (indeed, it constructs u by better and better approximations depending only on F). It guarantees existence and uniqueness of solutions, together with their continuous dependence on the initial data. For PDEs, these three requirements form the notion of *well-posedness.*

Theorem 1.6 (Cauchy–Peano theorem[8]). *If the vector field* $F(t,u)$ *is merely continuous, then for every* (t_0, u_0) *there exists a neighbourhood of* t_0 *where the ODE admits at least one* C^1 *solution satisfying* $u(t_0) = u_0$. *The solution need not be unique.*

The proof proceeds by approximating the trajectory with polygonal lines (Euler method) and applying compactness theorems such as Arzelà-Ascoli. Conceptually is the finite dimensional (and for continuous functions) paradigm that underlies modern weak solution constructions in PDE.

Example 1.7 (Non-uniqueness). A typical example is

$$u'(t) = \sqrt{u(t)}, \qquad u(0) = 0,$$

which admits two types of solutions (Exercise 1.2(a)). Another example given by Peano is

$$u'(t) = \frac{4u(t)t^3}{u(t)^2 + t^4}, \qquad u(0) = 0,$$

which admits five distinct solution types (Exercise 1.3(b)).

These theorems illustrate the delicate role of regularity the vector field F: higher regularity facilitates uniqueness.

Local vs global solutions The previous theorems establish only *local* results. It is natural to ask how far a solution can be extended in time. If it is defined for all $t \geq t_0$, it is called a *global solution.*

Example 1.8. (A blow-up example) Even when F is continuous and locally Lipschitz in u, the solution can blow up in finite time. A classical example is (Exercise 1.2(c))

$$u'(t) = u(t)^2, \qquad u(0) = u_0 > 0.$$

By contrast, changing the sign of the nonlinearity, $u'(t) = -u(t)^2$ admits a global solution. Thus not only the modulus of the nonlinearity, but also its sign matters.

Criterion 1.9. A standard sufficient condition for ensuring global existence is the *uniform Lipschitz bound* in u: there exists $L \geq 0$ such that

$$|F(t,u) - F(t,v)| \leq L\,|u - v| \qquad \text{for all } t \text{ and } u, v.$$

This prevents blow-up and yields global solutions. Such a criterion rarely extends to PDEs, even linear ones, because their associated operators are typically unbounded.

[7] Appeared first in Cauchy's lectures at the École Polytechnique (1830s) for C^1 vector fields, later generalized by Lipschitz, and finally given its modern proof via fixed points by Picard and Lindelöf.

[8] Published in 1890 by Peano as an extension of Cauchy's theorem.

Criterion 1.10. When F is continuous and locally Lipschitz in u with the *linear growth* estimate

$$|F(t,u)| \leq C\,(1+|u|) \qquad \text{for some } C>0 \text{ and all } (t,u), \tag{1.2}$$

then the solution is global (prove it). This already shows that a linear growth bound strongly influences uniqueness.

Example 1.11. For any initial data $u(0)=u_0 \in \mathbb{R}$, both

$$u'(t) = \sin(u(t)), \qquad u'(t) = \sin\left(u(t)^2\right),$$

admit global solutions (use the above criteria).

1.3.2 Well-Posedness in the sense of Hadamard

Early work on ODEs emphasized explicit integration (quadratures and special functions), but no general “finite formula" exists for solution to an arbitrary equations. The next step was looking for “countable formula" with infinite series, which is what Cauchy did in the first theorem above. However, this is limited to the local and analytic setting. The conceptual breakthrough still by Cauchy (the second theorem above) consist of constructing solutions through approximation and limiting processes (using the completeness of the real line) and then check uniqueness, given suitable boundary conditions, by a contraction estimate. Peano then extended this framework using compactness to construct solutions under lower regularity for F (third theorem). In the absence of explicit solutions and given that “solving” now has a more abstract meaning, what is the connection between the equation considered and the underlying physics? In his 1902 paper, Hadamard proposes the concept of **well-posedness** to answer this question.

Definition 1.12 (Cauchy problem & Hadamard well-posedness)**.** A Cauchy problem consists of a PDE (1.1) on a domain $\mathcal{U}$ together with boundary conditions, i.e. prescribed values of the unknown and possibly some of its derivatives on part of $\mathcal{U}$. It is *well-posed in a function space* X (in the sense of Hadamard) (e.g. $C^k(\mathcal{U})$, $H^k(\mathcal{U})$,…) if:

(i) There exists $u \in X$ solving (1.1) with the given boundary data;

(ii) The solution is unique in X, given the boundary data;

(iii) The solution depends continuously on the boundary data (in a suitable topology).

Remark 1.13.

1. (Meaning of well-posedness) Existence prevents over-determination; uniqueness prevents under-determination. For evolution problems, (i)–(ii) encode causality: the present data determine uniquely the future. (iii) means that causality has to behave continuously to be practical: we need that small data errors produce only small changes in the solution. The concept of well-posedness can be considered as a minimal requirement for physical consistency, and it is also useful from a modeling point of view to identify correct equations and boundary conditions.
2. (The choice of X) The function space X (together with a suitable topology) should be both large enough to find at least a solution and small enough (e.g. more regularity, more decay at infinity) to have uniqueness. This is a crucial part of the problem: finding the correct balance between these two requirements. When the function space is so large that the derivatives of the unknown appearing in the PDE are not guaranteed to exist in classical sense, we talk about **weak solutions**. The correct space is sometimes suggested by key physical quantities (e.g. energy, entropy).

3. (What we can export to PDEs) The Cauchy-Kovalevskaya theorem has *some* extension to PDEs (see next chapter). Usually PDEs, even linear ones, lead to unbounded operators, so searching for an analogue to PDEs of Picard-Lindelöf theorem seems to naive.[9] The proof (through Gronwall lemma) of the growth condition criterion (1.2) for extending solutions globally corresponds to the idea of a **priori (or energy) estimates** in PDEs.

[9] However, the Hille-Yosida theory can be seen (at some extent) as a generalization for (some) linear unbounded operators in PDEs.

LECTURE 3

1.4 Linear and classes of nonlinear PDEs

Definition 1.14. We say that the PDE is **linear** when the vector field F is a linear function of u and its derivatives; Then (1.1) reduces to[10]

$$\sum_{\alpha:|\alpha|\leq k} a_\alpha(x)\partial^\alpha u(x) = f(x)$$

for some **coefficients** a_α and **source term** f. When $f \equiv 0$ the PDE is said to be *homogeneous*. We say that the PDE is **semilinear** when the vector field F is linear in the highest order derivatives; the PDE takes the form

$$\sum_{\alpha:|\alpha|=k} a_\alpha(x)\partial^\alpha u(x) + a_0\left[x, u, \nabla u, \dots, \nabla^{k-1}u\right] = f$$

where the coefficient a_0 can be nonlinear in all variables. We say that the PDE is **quasilinear** when the vector field F is linear in the highest order derivatives but with possible nonlinear dependency in the lower-order derivatives; the PDE takes the form

$$\sum_{\alpha:|\alpha|=k} a_\alpha\left[x, u, \nabla u, \dots, \nabla^{k-1}u\right]\partial^\alpha u(x) + a_0\left[x, u, \nabla u, \dots, \nabla^{k-1}u\right] = f.$$

We say that the PDE is **fully nonlinear** if it is not one of the above forms. When the PDE takes the form of an evolution problem and the vector field F does not depend on time t, the PDE is said to be **autonomous**, and a solution u which does not depend on time is said to be **stationary**.

Example 1.15. The following PDEs demonstrate each of the above forms: $\Delta u = 0$ is linear, $\Delta u = \left(\frac{\partial u}{\partial x_1}\right)^2$ is semilinear, $u\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}$ on $u = u(x,y)$ is quasilinear, and $\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 0$ on $u = u(x,y)$ is fully nonlinear.

Some concrete examples of PDEs in physics are: the compressible and incompressible *Euler equations*, the compressible and incompressible *Navier–Stokes equations* (e.g. in the simplest incompressible form $\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{0}$ with $\nabla\cdot\mathbf{u} = 0$; the question of global regularity vs finite-time blow-up from smooth initial data in three space dimensions is a Millennium Prize Problem), the *Maxwell equations* in electromagnetism, the (Maxwell)-*Boltzmann equation* in kinetic theory, the (Jeans)-*Vlasov equations* in plasma physics and galactic dynamics, the *Schrödinger equation* in quantum mechanics, the *Einstein equations* in general relativity.

Two concrete examples of PDEs arising from within mathematics are the *Cauchy–Riemann equations* $\partial_x u - \partial_y v = 0$ and $\partial_y u + \partial_x v = 0$ in complex analysis (which imply $\Delta u = \Delta v = 0$) and the *Ricci flow* $\partial_t g_{ij} = -2R_{ij}$ (where g_{ij} is the metric tensor and R_{ij} is the Ricci tensor) in differential geometry.

[10] We write $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$.

2 The Cauchy-Kovalevskaya Theorem

This section deals with the only "general" theorem for PDEs that can be carried over from ODEs, the Cauchy–Kovalevskaya Theorem. Its rigorous formulation introduces the notions of non-characteristic data, principal symbol, and the basic classification of PDEs. However, analyticity is often not a satisfying functional setting for PDEs.

2.1 Real Analyticity

Definition 2.1. Given $\mathcal{U} \subset \mathbb{R}^n$ open non-empty, a function $f : \mathcal{U} \to \mathbb{R}$ is **real analytic near** $\tilde{x} \in \mathcal{U}$ if there is $r > 0$ and real constants $(f_\alpha)_{\alpha \in \mathbb{N}^n}$ so that the series

$$\sum_{\alpha \in \mathbb{N}^n} f_\alpha (x - \tilde{x})^\alpha \tag{2.1}$$

(where we denote $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$) is converging, respectively **absolutely** converging[11] when $n \geq 2$, and converges to $f(x)$ for $x \in B(\tilde{x}, r)$. A function $f : \mathcal{U} \to \mathbb{R}$ is **real analytic on** $\mathcal{U}$ if it is real analytic near any $\tilde{x} \in \mathcal{U}$. A vector valued function $f : \mathcal{U} \to \mathbb{R}^m$ ($m \geq 2$) is real analytic if each of its component is real analytic. The set of all real analytic functions on a given open set $\mathcal{U}$ is denoted by $C^\omega(\mathcal{U})$.

Remark 2.2. Simple examples of analytic functions are polynomials, the exponential, cosinus and sinus. The complex conjugate $z \mapsto \bar{z}$ is not complex analytic but is real analytic on $\mathbb{R}^2$. The function e^{-1/x^2} extended by 0 at $x = 0$ is smooth but not real analytic near $x = 0$ (its Taylor series is zero). More generally non-zero smooth compactly supported functions are not real analytic. The Liouville theorem (in complex analysis) is false for real analytic functions (e.g. $f(x) = 1/(1 + x^2)$ on $\mathbb{R}$ is bounded and non-constant). Note finally that real analyticity is a local property: f is real analytic near $\tilde{x}$ implies that f is real analytic on a neighbourhood of $\tilde{x}$ (see Exercise 1.8(v)).

Proposition 2.3. *Given $\mathcal{U} \subset \mathbb{R}^n$ open, a function $f : \mathcal{U} \to \mathbb{R}$ is real analytic on $\mathcal{U}$ if and only if $f \in C^\infty(\mathcal{U})$ and for any compact $K \subset \mathcal{U}$ there are constants $C(K), r(K) > 0$ so that*

$$\forall x \in K, \ \forall \alpha \in \mathbb{N}^n, \quad |\partial^\alpha f(x)| \leq C(K) \, \frac{\alpha!}{r(K)^{|\alpha|}},$$

where $\alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Remark 2.4. Another equivalent definition is the following: for every $a \in \mathcal{U} \subset \mathbb{R}^n$, there exists an open neighborhood $\mathcal{V} \subset \mathbb{C}^n$ of a and a holomorphic function $F : \mathcal{V} \to \mathbb{C}$ such that $F\big|_{\mathcal{V} \cap \mathbb{R}^n} = f$ (prove it in case $\mathcal{U} \subset \mathbb{R}$). When $\mathcal{U} = \mathbb{R}^n$, real analyticity can be characterized by the exponential decay of the Fourier transform (we will not use this characterization).

[11] Alternatively, in the definition we can just ask for conditional convergence but we need to clarify the way we sum on multiindices: $\sum_{k=0} \sum_{\alpha : |\alpha = k|} \sum_{\alpha \in \mathbb{N}^n} f_\alpha (x - \tilde{x})^\alpha$.

LECTURE 4

Proof of Proposition 2.3. Preliminaries. Let $r > 0$ be such that the power series (2.1) converges absolutely on $B(\tilde{x}, r)$.[12] Then for any $\tilde{r} \in (0, r/\sqrt{n})$, setting

$$C(\tilde{r}) := \sum_{\alpha \in \mathbb{N}^n} |f_\alpha| \, \tilde{r}^{|\alpha|} < \infty,$$

which is finite by evaluating the series at $\tilde{x}+h$ where $h = (\tilde{r}, \dots, \tilde{r})$ (so that $|h - \tilde{x}| = \sqrt{n}\, \tilde{r} < r$), we have

$$|f_\alpha| \le C(\tilde{r})\, \tilde{r}^{-|\alpha|} \qquad \forall\, \alpha \in \mathbb{N}^n. \tag{2.2}$$

Consequently, for any $0 < \hat{r} < \tilde{r}$ the series is absolutely and uniformly convergent on $\overline{B(\tilde{x}, \hat{r})}$ since

$$\sum_\alpha |f_\alpha| \, |x - \tilde{x}|^{|\alpha|} \le C(\tilde{r}) \sum_\alpha \left(\tfrac{|x - \tilde{x}|}{\tilde{r}} \right)^{|\alpha|} < \infty.$$

Proof of $\Longrightarrow$. Assume f is real analytic near $\tilde{x}$, i.e. (2.1) holds and the series is convergent (absolutely convergent for $|x - \tilde{x}| < r$ when $n \ge 2$). Fix numbers

$$0 < \bar{r} < \hat{r} < \tilde{r} < r/\sqrt{n} \le r,$$

so that (2.2) holds with $\tilde{r}$, and the uniform convergence discussion applies on $\overline{B(\tilde{x}, \hat{r})}$.

Then the series and all its termwise derivatives converge uniformly on $\overline{B(\tilde{x}, \hat{r})}$, hence $f \in C^\infty(B(\tilde{x}, \hat{r}))$ and

$$\partial^\alpha f(\tilde{x}) = \alpha!\, f_\alpha.$$

Now take $x \in \overline{B(\tilde{x}, \bar{r})}$ and any $\beta \in \mathbb{N}^n$. Then

$$\begin{aligned} |\partial^\beta f(x)| &\le \sum_{\alpha \ge \beta} |f_\alpha| \, \frac{\alpha!}{(\alpha - \beta)!} \, |x - \tilde{x}|^{|\alpha - \beta|} \\ &\le C(\tilde{r}) \sum_{\alpha \ge \beta} \tilde{r}^{-|\alpha|} \frac{\alpha!}{(\alpha - \beta)!} \, \bar{r}^{|\alpha - \beta|} \\ &= C(\tilde{r})\, \tilde{r}^{-|\beta|} \sum_{\alpha \ge \beta} \frac{\alpha!}{(\alpha - \beta)!} \left(\tfrac{\bar{r}}{\tilde{r}} \right)^{|\alpha - \beta|}. \end{aligned}$$

Using the multiindex identity (see Exercise 1.7)

$$\sum_{\alpha \ge \beta} \frac{\alpha!}{(\alpha - \beta)!} \lambda^{|\alpha - \beta|} = \beta!\, (1 - \lambda)^{-(|\beta| + n)} \qquad (0 < \lambda < 1),$$

with $\lambda = \bar{r}/\tilde{r}$, we get

$$|\partial^\beta f(x)| \le C(\tilde{r})\, \tilde{r}^{-|\beta|}\, \beta!\, (1 - \bar{r}/\tilde{r})^{-(|\beta| + n)} = \frac{C(\tilde{r})}{(1 - \bar{r}/\tilde{r})^n}\, \beta!\, (\tilde{r} - \bar{r})^{-|\beta|}.$$

Thus on each closed ball $\overline{B(\tilde{x}, \bar{r})}$ we have the desired bound. Covering any compact $K \subset \mathcal{U}$ by finitely many such closed balls yields the global constants $C(K), r(K) \in (0, \infty)$, obtaining $\left|\partial^\beta f(x)\right| \le C(K) \beta! r(K)^{-|\beta|}$ for all $x \in K$.

[12] In general, we define the radius of convergence $r \in [0, \infty]$ as the supremum of the radii where the series is absolutely converging.

Proof of $\Longleftarrow$. Fix $\tilde{x} \in \mathcal{U}$ and let $K := \overline{B(\tilde{x}, \rho)} \subset \mathcal{U}$. By assumption, there exist $\tilde{C}, \tilde{r} > 0$ such that

$$|\partial^\alpha f(x)| \leq \tilde{C}\, \frac{\alpha!}{\tilde{r}^{|\alpha|}} \qquad \text{for all } x \in K,\ \alpha \in \mathbb{N}^n.$$

Set $\rho_* := \min\{\rho, \tilde{r}/2\}$. For any x with $|x - \tilde{x}| < \rho_*$, the segment $\tilde{x} + t(x - \tilde{x})$, $t \in [0,1]$, is contained in K, so we may apply Taylor's formula with integral remainder:

$$f(x) = \sum_{|\alpha| \leq k} \partial^\alpha f(\tilde{x}) \frac{(x-\tilde{x})^\alpha}{\alpha!} + \sum_{|\alpha|=k+1} R_\alpha(x)\,(x-\tilde{x})^\alpha,$$

where

$$R_\alpha(x) := \frac{|\alpha|}{\alpha!} \int_0^1 (1-t)^{|\alpha|-1} \partial^\alpha f(\tilde{x} + t(x-\tilde{x}))\ \mathrm{d}t.$$

Thanks to the controls on the $\partial^\alpha f$'s we have $\sum_{\alpha \in \mathbb{N}^n} \left| \frac{\partial^\alpha f(\tilde{x})}{\alpha!} (x-\tilde{x})^\alpha \right| < \infty$ (prove it) so Taylor series is absolutely converging on $B(\tilde{x}, \rho_*)$. Using the bound on $\partial^\alpha f$ we obtain

$$|R_\alpha(x)| \leq \tilde{C}\, \tilde{r}^{-|\alpha|}.$$

Hence

$$\Big| \sum_{|\alpha|=k+1} R_\alpha(x)\,(x-\tilde{x})^\alpha \Big| \leq \tilde{C}\, \tilde{r}^{-(k+1)} \sum_{|\alpha|=k+1} |(x-\tilde{x})^\alpha|.$$

Since $\sum_{|\alpha|=m} |y^\alpha| \leq C_n\, |y|^m$ for some constant $C_n > 0$, we obtain (prove it)

$$\Big| \sum_{|\alpha|=k+1} R_\alpha(x)\,(x-\tilde{x})^\alpha \Big| \leq C''(k+n)^{n-1}\, \tilde{r}^{-(k+1)} |x-\tilde{x}|^{k+1}.$$

Since $|x - \tilde{x}| \leq \rho_*$ we get

$$\Big| \sum_{|\alpha|=k+1} R_\alpha(x)\,(x-\tilde{x})^\alpha \Big| \leq C''(k+n)^{n-1} \Big(\frac{|x-\tilde{x}|}{\tilde{r}}\Big)^{k+1} \leq C''(k+n)^{n-1}\, 2^{-(k+1)} \xrightarrow{k\to\infty} 0.$$

Therefore the remainder tends to zero, and the Taylor series converges absolutely to $f(x)$ on $B(\tilde{x}, \rho_*)$. Since $\tilde{x}$ was arbitrary, f is real analytic on $\mathcal{U}$. □

The main idea in part (⇒) of the proof above consists of a comparison between intricate combinatorial sums and the derivatives of the geometric series. This comparison is the core of Cauchy's argument in the Cauchy–Kovalevskaya theorem (for ODEs and later for PDEs). It motivates the following definition.

Definition 2.5. Let $f = \sum_{\alpha \geq 0} f_\alpha x^\alpha$ and $g = \sum_{\alpha \geq 0} g_\alpha x^\alpha$ be two formal power series. We say that g **majorizes** f, or g is a **majorant** of f, written $g \gg f$, if $g_\alpha \geq |f_\alpha|$ for all $\alpha \geq 0$. If f and g are valued in $\mathbb{R}^m$, then we say that g majorizes f if $g_j \gg f_j$ for all $j = 1, \dots, m$.

Remark 2.6. Note that $g \gg f$ implies $g_\alpha \geq 0$ for all $\alpha \in \mathbb{N}^n$.

Proposition 2.7. *Let f and g be formal power series.*

(i) If $g \gg f$ and g converges for $|x| < r$, then f converges absolutely for $|x| < r$ as well.

(ii) If f converges absolutely for $|x| < r$ and $\tilde{r} \in (0, r/n)$, then there exists a majorant $\bar{f} \gg f$ which converges on $|x| < \tilde{r}$.

Proof. (i) Fix $x \in B(0,r)$ and set $y := (|x_1|,\dots,|x_n|)$. Then $|y| = |x| < r$, and for the truncated sums

$$S_k = \sum_{|\alpha|\le k} \left|f_\alpha x^\alpha\right| = \sum_{|\alpha|\le k} |f_\alpha|\, y^\alpha \le \sum_{|\alpha|\le k} g_\alpha\, y^\alpha \le \sum_{\alpha\ge 0} g_\alpha\, y^\alpha = g(y),$$

where we used that the g_α's are non-negative. Since $g(y) < \infty$ then S_k is uniformly bounded (and monotone) in k. Therefore S_k converges, i.e. f converges absolutely at x.

(ii) Fix $\tilde r \in (0, r/n)$ and choose $\hat r$ with

$$\sqrt{n}\,\tilde r \;<\; \hat r \;<\; \frac{r}{\sqrt{n}}.$$

Let $y = (\hat r, \dots, \hat r)$; then $|y| = \hat r\sqrt{n} < r$, so by absolute convergence

$$S \;:=\; \sum_{\alpha\ge 0} |f_\alpha|\,\hat r^{|\alpha|} \;<\; \infty, \qquad \text{hence} \qquad |f_\alpha| \;\le\; S\,\hat r^{-|\alpha|} \quad \text{for all } \alpha.$$

We construct two examples of majorants.
Majorant 1. Define

$$\bar f(x) \;:=\; S\prod_{i=1}^{n} \frac{1}{1 - x_i/\hat r} \;=\; S\sum_{\alpha\ge 0} \hat r^{-|\alpha|} x^\alpha.$$

Then $\bar f_\alpha = S\,\hat r^{-|\alpha|} \ge |f_\alpha|$, so $\bar f \gg f$. Moreover, the product converges whenever $|x_i| < \hat r$ for all i. Since $|x| < \tilde r$ implies $|x_i| \le |x| < \tilde r < \hat r$, it also converges on the ball $B(0, \tilde r)$.
Majorant 2.[13] We define

$$\tilde f(x) \;:=\; \frac{S\hat r}{\hat r - (x_1 + \cdots + x_n)} \;=\; S\sum_{k\ge 0}\left(\frac{x_1 + \cdots + x_n}{\hat r}\right)^k \;=\; S\sum_{\alpha\ge 0} \frac{|\alpha|!}{\alpha!\,\hat r^{|\alpha|}}\, x^\alpha,$$

where the last is a multiindex identity (see Exercise 1.7). Its coefficients satisfy

$$\tilde f_\alpha \;=\; S\,\frac{|\alpha|!}{\alpha!\,\hat r^{|\alpha|}} \;\ge\; S\,\hat r^{-|\alpha|} \;\ge\; |f_\alpha|,$$

so $\tilde f \gg f$. For convergence

$$\sum_{\alpha\ge 0}\left|\tilde f_\alpha x^\alpha\right| = S\sum_{k\ge 0}\frac{1}{\hat r^k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\,|x|^\alpha = S\sum_{k\ge 0}\left(\frac{|x_1| + \cdots + |x_n|}{\hat r}\right)^k.$$

If $|x| < \tilde r$, then $\sum_{i=1}^n |x_i| \le \sqrt{n}\,|x| < \sqrt{n}\,\tilde r < \hat r$, so the last geometric series converges. Hence $\tilde f$ converges on $|x| < \tilde r$.[14] □

[13]The two majorants coincide in the scalar case $n = 1$.
[14]Majorant 1 requires only $\tilde r \in (0, r/\sqrt{n})$, while majorant 2 requires $\tilde r \in (0, r/n)$.

LECTURE 5

2.2 The Cauchy-Kovalevskaya theorem for ODEs

Theorem 2.8 (Cauchy-Kovalevskaya theorem for (scalar autonomous) ODEs)**.** *Let $a, b > 0$, $u_0 \in \mathbb{R}$, and $F : (u_0 - b, u_0 + b) \to \mathbb{R}$ be real analytic. If $u : (-a, a) \to (u_0 - b, u_0 + b)$ is a C^1 solution of*

$$u'(t) = F(u(t)), \qquad u(0) = u_0,$$

then u is real analytic on $(-a, a)$.

Remark 2.9. As stated, the construction of solutions is already settled by the Picard-Lindelöf theorem so this is a *regularity theorem*: the solution is real analytic in the region where the field F is analytic. Instead in the analogue theorem for PDEs, the construction of the solution is part of the theorem. To prove the theorem, we can assume $u_0 = 0$. It is also sufficient to prove analyticity near $t = 0$, as then one can apply the same argument starting with any $t_0 \in (-a, a)$.

We give four different proofs to show several ideas. Only the proof by majorants will extend to the PDE case.

Proof by Picard iterations in $\mathbb{C}$ (Not examinable). Since F is real analytic at u_0, there exists $\rho > 0$ and a holomorphic extension (still called F) to the complex disc $D_\rho(u_0)$. Choose $0 < \rho_1 < \rho$ so that $\overline{D_{\rho_1}(u_0)} \subset D_\rho(u_0)$ and set $M := \sup_{|w-u_0|\le\rho_1} |F(w)| < \infty$, $L := \sup_{|w-u_0|\le\rho_1} |F'(w)| < \infty$. Pick $R > 0$ with $MR \le \rho_1/2$ and $q := LR < 1$, and consider the closed disk $\overline{D}_R$.
Define $u_0(z) \equiv u_0$ and, for $n \ge 0$,

$$u_{n+1}(z) := u_0 + \int_0^z F(u_n(z'))\, dz' = u_0 + \int_0^1 F\big(u_n(tz)\big)\, z\, dt, \qquad |z| \le R.$$

By induction, each u_n is holomorphic on D_R, $u_n(0) = u_0$, and $|u_{n+1}(z) - u_0| \le MR \le \frac{\rho_1}{2}$, so $u_n(\overline{D}_R) \subset \overline{D_{\rho_1}(u_0)}$.
For $|z| \le R$,

$$|u_{n+1}(z)-u_n(z)| \le \int_0^1 |F(u_n(tz))-F(u_{n-1}(tz))|\, |z|\, dt \le LR\, \|u_n-u_{n-1}\|_\infty = q\, \|u_n-u_{n-1}\|_\infty.$$

Hence $\|u_m - u_\ell\|_\infty \to 0$ for $m \ge \ell$ and $\ell \to \infty$. Thus $(u_n)_n$ is Cauchy in the Banach space $(C(\overline{D}_R), \|\cdot\|_\infty)$ (the space of continuous functions on $\overline{D}_R$) and by completeness there exist $u \in C(\overline{D}_R)$ with $u_n \to u$ uniformly on $\overline{D}_R$.
The limit is holomorphic. Indeed, for any closed piecewise C^1 curve $\gamma \subset D_R$, Cauchy's integral theorem gives $\int_\gamma u_n\, dz = 0$ for all n, and by the uniform limit also $\int_\gamma u\, dz = 0$; by Morera's theorem, u is holomorphic on D_R. Passing to the limit in the defining integrals gives

$$u(z) = u_0 + \int_0^z F\big(u(z')\big)\, dz', \qquad z \in D_R,$$

so for z real in D_R we have $u'(z) = F(u(z))$ and $u(0) = u_0$, thus by uniqueness it coincides with the given C^1 solution, which then it is analytic (as it extends as a holomorphic function).
□

Proof by separation of variables in ODE theory (not examinable). In case $F(0) = 0$ then the solution is $u \equiv 0$ that is real analytic (and by uniqueness coincides with the given C^1 solution). Letting $F(0) \neq 0$ we choose $b' \in (0, b)$ such that $F \neq 0$ on $(-b', b')$. Then $1/F$ is analytic there and we define

$$G(y) := \int_0^y \frac{1}{F(x)}\, dx, \qquad y \in (-b', b'),$$

so G is analytic and $G'(0) = 1/F(0) \neq 0$. For some $a' \in (0, a)$, $u((-a', a')) \subset (-b', b')$ and by the chain rule

$$\frac{d}{dt}\, G(u(t)) = \frac{1}{F(u(t))}\, u'(t) = 1,$$

hence $G(u(t)) = t$ for $|t| < a'$ (using $G(0) = 0$). By the *analytic inverse function theorem*, G^{-1} is analytic near 0, so $u(t) = G^{-1}(t)$ is analytic near 0. □

Remark 2.10. This proof secretly relies on either complex extension or the method of majorants. Indeed, the proofs of the analytic inverse function theorem proceed by locally complexifying real-analytic maps and applying the holomorphic implicit function theorem (proved via Cauchy's integral formula), or otherwise by series reversion (via recurrence relation of coefficients or Lagrange inversion theorem), with convergence ensured by Cauchy majorants.[15]

Proof by embedding the ODE in a one-parameter family of ODEs (not examinable). Fix $R \in (0, b)$. Since F is real-analytic, there exists $\rho > 0$ and a holomorphic extension (still denoted by F) to the tube $\Omega := \{\, w = x + iy \in \mathbb{C} : |x| \leq R,\ |y| < \rho \,\}$. Set $M := \sup_{w\in\Omega} |F(w)|$ and $L := \sup_{w\in\Omega} |F'(w)|$.
For $z \in \mathbb{C}$ consider

$$u_z'(t) = z\, F(u_z(t)), \qquad u_z(0) = 0, \qquad \text{for} \quad |z| \leq 2. \tag{2.3}$$

On Ω, $|zF(w)| \leq 2M$ and the Lipschitz constant of $w \mapsto zF(w)$ is bounded by $2L$. Choose

$$\tau \leq \min\left\{\frac{R}{2M},\ \frac{\rho}{2M},\ \frac{1}{2L}\right\}. \tag{2.4}$$

By Picard–Lindelöf there is a unique $u_z \in C^1([-\tau, \tau])$ with $u_z(t) \in \Omega$ for $|t| \leq \tau$, with τ independent of z (check how the interval of existence is estimated in the proof of Picard-Linderöf).
Let $u := u_1$. For real $z \in (-2, 2)$, the map $t \mapsto u(zt)$ solves (2.3); by uniqueness,

$$u_z(t) = u(zt) \qquad \text{for} \quad |t| \leq \tau,\ z \in (-2, 2) \cap \mathbb{R}. \tag{2.5}$$

We show that $u_z(t)$ is holomorphic in z. Since the map $z \mapsto u_z(t)$ is C^1 in the real sense, that is in the variables $\Re(z), \Im(z)$ (check it), then to show that is holomorphic it is enough to check that $\partial_{\bar z} u_z \equiv 0$ (this is equivalent to the Cauchy-Riemann equations), where $\partial_{\bar z} := \frac{1}{2}(\partial_x + i\partial_y)$. Differentiate (2.3) in $\bar z$. Using $\partial_{\bar z} z = 0$ and the holomorphic chain rule on Ω,

$$\partial_t(\partial_{\bar z} u_z) = z\, F'(u_z)\, \partial_{\bar z} u_z, \qquad \partial_{\bar z} u_z(0) = 0,$$

[15] Historically, Cauchy is credited with the first proofs of the implicit function theorem, one employing holomorphic functions and another based on his method of majorants, see Krantz–Parks, *The Implicit Function Theorem* (2002).

hence $\partial_{\bar z} u_z \equiv 0$. Therefore, for fixed t, $z \mapsto u_z(t)$ is holomorphic on $|z| < 2$.
Thus, for fixed t we have

$$u_z(t) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \, \partial_z^\ell u_z(t)\big|_{z=0} \, z^\ell, \qquad |z| < 2.$$

At $z = 1$,

$$u(t) = u_1(t) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \, \partial_z^\ell u_z(t)\big|_{z=0}.$$

By (2.5), for real z near 0, $u_z(t) = u(zt)$, hence

$$\partial_z^\ell u_z(t)\big|_{z=0} = \frac{d^\ell}{dz^\ell} u(zt)\Big|_{z=0} = t^\ell \, u^{(\ell)}(0).$$

Therefore,

$$u(t) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \, u^{(\ell)}(0) \qquad \text{for} \quad |t| < \tau,$$

and u is real analytic near 0. □

Proof by the method of majorants. Let u be the given C^1 solution of $u' = F(u(t))$, with F analytic. Then $F(u(t))$ is C^1, hence u' is C^1 and u is C^2. Iterating this, we conclude $u \in C^\infty$. We can compute the higher derivatives inductively:

$$\begin{cases} u^{(1)}(t) = F^{(0)}(u(t)), \\ u^{(2)}(t) = F^{(1)}(u(t)) \, u^{(1)}(t) = F^{(1)}(u(t)) \, F^{(0)}(u(t)), \\ u^{(3)}(t) = F^{(2)}(u(t)) \big(F^{(0)}(u(t))\big)^2 + \big(F^{(1)}(u(t))\big)^2 F^{(0)}(u(t)), \\ \dots \end{cases}$$

By induction, $u^{(n)}$ is always a polynomial p_n in the values $F^{(k)}(u)$, for $k = 0, \dots, n-1$, with non-negative integer coefficients. Moreover, this inductive structure is universal: the polynomials do not depend on the specific choice of F. In particular

$$u^{(n)}(0) = p_n\big(F^{(0)}(0), F^{(1)}(0), \dots, F^{(n-1)}(0)\big) \qquad \text{for} \quad n \geq 1.$$

Define the formal Taylor series at 0 by

$$\widehat{u}(t) := \sum_{n \geq 0} \frac{p_n(F^{(0)}(0), \dots, F^{(n-1)}(0))}{n!} \, t^n = \sum_{n \geq 0} \frac{u^{(n)}(0)}{n!} \, t^n, \qquad p_0 := u^{(0)}(0) := u_0 = 0.$$

Assume there exists an analytic G majorant of F at 0, i.e. such that

$$G^{(k)}(0) \geq \big|F^{(k)}(0)\big| \qquad \forall k \geq 0,$$

and let v solve

$$v'(t) = G(v(t)), \qquad v(0) = 0,$$

analytic on $|t| < R$. The same differentiation scheme gives

$$v^{(n)}(0) = p_n\big(G^{(0)}(0), G^{(1)}(0), \dots, G^{(n-1)}(0)\big).$$

Since the p_n have non-negative coefficients

$$\left|u^{(n)}(0)\right| \le p_n\big(\,|F^{(0)}(0)|,\dots,|F^{(n-1)}(0)|\,\big) \le p_n\big(G^{(0)}(0),\dots,G^{(n-1)}(0)\big) = v^{(n)}(0).$$

Because v is analytic on $|t| < R$, its Taylor series converges there; this implies the that Taylor series for $\widehat{u}$ also converges for $|t| < R$ (Proposition 2.7(i)), so $\widehat{u}$ is analytic near 0. Moreover, $\hat{u}'(t)$ and $F(\hat{u}(t))$ are both analytic (by composition of analytic functions) and all their derivatives agree at 0, thus it solves the ODE near 0. By local uniqueness $u = \widehat{u}$ near 0, so u is real analytic near 0.
Since F is real analytic at 0 there exist $C, r > 0$ with

$$|F^{(k)}(0)| \le C\,k!\,r^{-k} \qquad \forall k \ge 0.$$

Consider the majorant constructed in the proof of Proposition 2.7(ii)

$$G(x) := \sum_{k\ge 0} C\,r^{-k}x^k = \frac{Cr}{r-x} \quad (|x| < r),$$

so $G^{(k)}(0) = C\,k!\,r^{-k} \ge |F^{(k)}(0)|$. The initial value problem

$$v'(t) = G(v(t)), \qquad v(0) = 0$$

has the explicit solution

$$v(t) = r - r\sqrt{1 - \frac{2Ct}{r}},$$

which is analytic for $|t| < R$ with $R = \frac{r}{2C}$. This provides the required G and v, completing the proof near 0. $\square$

Remark 2.11 (Cauchy–Kovalevskaya for ODE systems). As in the proof of Proposition 2.7(ii), take (for example) the symmetric vector majorant $\bar{f} = G = (G_1, \dots, G_m)$ with $G_1 = \dots = G_m = \frac{Cr}{r-(x_1+\dots+x_m)}$. With this choice, the scalar majorant proof of Theorem 2.8 can be carried on to obtain the analogue result for ODE system: if $F : B(u_0, b) \to \mathbb{R}^m$ is real analytic and $u : (-a, a) \to B(u_0, b)$ is a C^1 solution of $u'(t) = F(u(t))$ with $u(0) = u_0$, then u is real analytic (left as an exercise).

Remark 2.12 (Non-autonomous case). The case of non-autonomous $F = F(t, u)$ can be reduce to an autonomous system: $y = (y_0, y_1, \dots, y_m) = (t, u)$ and $\dot{y} = \tilde{F}(y) := (1, F(y_0, y_1, \dots, y_m))$, $y(0) = (0, u_0)$. Thus Cauchy-Kovalevskaya applies in this case as well.

LECTURE 6

2.3 The Cauchy-Kovalevskaya theorem for PDEs

We consider a k-th order scalar quasilinear PDE[16]

$$\sum_{|\alpha|=k} a_\alpha \left(x, u, \nabla u, \ldots, \nabla^{k-1} u\right) \partial_x^\alpha u + a_0 \left(x, u, \nabla u, \ldots, \nabla^{k-1} u\right) = 0, \quad x \in \mathcal{U} \subset \mathbb{R}^n \tag{2.6}$$

on the open domain $\mathcal{U} \subset \mathbb{R}^n$. We want to extend the method of majorant to construct real analytic solutions.

Note that when $k = 1$ there is an alternative simpler proof of Cauchy-Kovalevskaya theorem, making use of the ODE result and based on the so-called method of characteristics[17]. However, it fails for systems of first-order PDEs (and thus also for scalar higher-order PDEs), so we need a more general proof.

2.3.1 Cauchy problem

We must first characterize the set on which conditions complementing the PDE (2.6) will be imposed.

Definition 2.13. Given $\mathcal{U} \subset \mathbb{R}^n$ open and nonempty, and $\Sigma \subset \mathcal{U}$, we say that Σ is a **smooth** (respectively **real analytic) hypersurface near** $x \in \Sigma$ if there exist $\varepsilon > 0$, an open set $\mathcal{V} \subset \mathbb{R}^n$, and a bijection

$$\Phi : B(x, \varepsilon) \to \mathcal{V}$$

such that Φ and Φ^{-1} are smooth (respectively real analytic), with $\Phi(x) = 0$, and

$$\Phi\big(\Sigma \cap B(x, \varepsilon)\big) = \{y_n = 0\} \cap \mathcal{V}.$$

Also, we say that Σ **is a smooth (respectively real analytic) hypersurface in** $\mathcal{U}$ if the previous property holds near every $x \in \Sigma$.

Remark 2.14. To connect to your differential geometry course, a hypersurface Σ is an immersed embedded smooth (resp. real analytic) submanifolds with codimension 1 without boundary. Real analytic submanifolds are defined like smooth submanifolds except that all local parametrisations are required to be real analytic.

Normal coordinates near a hypersurface. Let $\Sigma \subset \mathcal{U} \subset \mathbb{R}^n$ be a smooth (resp. real analytic) hypersurface, and let $x \in \Sigma$. Then there exist[18] :

- a smooth (resp. real analytic) unit normal vector $N : \Sigma \to \mathbb{S}^{n-1}$,
- a smooth (resp. real analytic) map

$$\Psi : \; B_{\mathbb{R}^{n-1}}(0, \varepsilon) \times (-\varepsilon, \varepsilon) \; \longrightarrow \; \mathcal{U}_x,$$

[16] We recall that, letting $\mathcal{U} \subset \mathbb{R}^\ell$ be open ($\ell \geq 2$ being the number of variables) and $u : \mathcal{U} \to \mathbb{R}$, for $j \in \mathbb{N}$ the j-th iterated gradient is $\nabla^j u := (\partial_{x_{i_1}} \cdots \partial_{x_{i_j}} u)_{1 \leq i_1, \ldots, i_j \leq \ell}$, and for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{N}^\ell$ we write $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_\ell}^{\alpha_\ell}$.

[17] We will discuss such method for hyperbolic PDEs in Chapter 5.

[18] We assume without proof these properties that follow from the definition of hypersurface.

such that, writing $\tilde{y} = (y_1, \ldots, y_{n-1})$ and $y_n \in \mathbb{R}$,

$$\Psi(\tilde{y}, y_n) \;=\; \tilde{\Psi}(\tilde{y}) \;+\; y_n\, N\big(\tilde{\Psi}(\tilde{y})\big), \qquad \Psi(0,0) = x, \tag{2.7}$$

where $\tilde{\Psi} : B_{\mathbb{R}^{n-1}}(0, \varepsilon) \to \Sigma \cap \mathcal{U}_x \subset \mathcal{U} \subset \mathbb{R}^n$ is smooth (resp. real analytic), and $\mathcal{U}_x \subset \mathcal{U}$ is a neighborhood of x. Differentiating we get

$$\partial_{y_n} \Psi(y) = N(\tilde{\Psi}(\tilde{y})).$$

For the tangential directions, fix $x' \in \mathcal{U}_x \cap \Sigma$, and choose $\tilde{y}'$ such that $\Psi(\tilde{y}', 0) = x'$. Then the tangent space to Σ at x' is

$$T_{x'}\Sigma \;=\; \operatorname{Span}\left(\partial_{y_1}\Psi(\tilde{y}', 0), \ldots, \partial_{y_{n-1}}\Psi(\tilde{y}', 0)\right).$$

Now consider the chart $\Phi := \Psi^{-1}$ associated with this parametrisation. Let

$$\varphi := \Phi_n : \mathcal{U}_x \to \mathbb{R}$$

be its last component. By construction $\Sigma \cap \mathcal{U}_x = \{\varphi = 0\} \cap \mathcal{U}_x$, and φ is smooth (resp. real analytic).
We also claim that $\nabla_x \varphi(x') = N(x')$ for all $x' \in \mathcal{U}_x \cap \Sigma$. Indeed, by definition of φ we have the identity $\varphi(\Psi(y)) = y_n$ for all y in the domain. Differentiating this with respect to y_i for $i = 1, \ldots, n-1$ gives

$$\nabla_x \varphi(\Psi(y)) \cdot \partial_{y_i} \Psi(y) = 0.$$

Thus $\nabla_x \varphi(\Psi(y))$ is orthogonal to each $\partial_{y_i}\Psi(y)$, hence orthogonal to the tangent space of Σ at $\Psi(y)$, and therefore collinear with $N(\Psi(y))$. Differentiating instead with respect to y_n gives

$$\nabla_x \varphi(\Psi(y)) \cdot \partial_{y_n} \Psi(y) = \nabla_x \varphi(\Psi(y)) \cdot N(\tilde{\Psi}(\tilde{y})) = 1.$$

Since $\nabla_x \varphi(\Psi(y))$ is collinear with $N(\Psi(y))$ and their dot product is 1, they must agree. In particular, for $x' = \Psi(\tilde{y}', 0) \in \Sigma$, we have $\nabla_x \varphi(x') = N(x')$.

To prescribe the conditions on Σ complementing equation (2.6), we will need derivatives of u in the normal direction.

Definition 2.15. Let $\Sigma \subset \mathcal{U} \subset \mathbb{R}^n$ be a smooth (resp. real analytic) hypersurface, let $N : \Sigma \to \mathbb{R}^n$ be the corresponding smooth (resp. real analytic) unit normal, and let $u \in C^j(\mathcal{U})$. For $x \in \Sigma$ and $j \in \mathbb{N}$, we define the j**-th normal derivative** of u at x by

$$\partial_N^j u(x) := \sum_{|\alpha| = j} \partial_x^\alpha u(x)\, N(x)^\alpha = \sum_{\alpha_1 + \cdots + \alpha_n = j} \frac{\partial^j u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}(x)\, N_1(x)^{\alpha_1} \ldots N_n(x)^{\alpha_n}.$$

Now, we prescribe on a hypersurface Σ the value of u together with its first $k-1$ normal derivatives. Informally, this specifies how the solution starts to extend away from Σ. This motivates the following definition.

Definition 2.16. Let $\Sigma \subset \mathcal{U} \subset \mathbb{R}^n$ be a smooth (resp. real analytic) hypersurface, and let $g_0, \ldots, g_{k-1} : \Sigma \to \mathbb{R}$ be smooth (resp. real analytic) functions. The **Cauchy problem** for (2.6) with Cauchy data $g_0, \ldots, g_{k-1}$ on Σ is the problem of finding a function u solving (2.6) in an open set $V \subset \mathcal{U}$ such that $\Sigma \cap V \neq \emptyset$, and

$$u = g_0, \quad \partial_N u = g_1, \quad \partial_N^2 u = g_2, \; \ldots, \; \partial_N^{k-1} u = g_{k-1} \quad \text{on } \Sigma \cap V. \tag{2.8}$$

We call Σ the **Cauchy hypersurface** and $\{g_j\}_{j=0}^{k-1}$ the **Cauchy data**.

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Remark 2.17 (Are (2.8) "boundary conditions"?). We prescribe data on a hypersurface $\Sigma \subset \mathcal{U}$, which in general is *not* the topological boundary $\partial\mathcal{U}$. We will still sometimes call these "boundary conditions". The reason is that near any point, $\Sigma = \{\varphi = 0\}$ splits a neighbourhood into the two sides $\{\varphi > 0\}$ and $\{\varphi < 0\}$, and Σ is their common boundary. For instance, we will see that for the wave equation $u_{tt} - \Delta_x u = 0$ on $\mathcal{U} = \mathbb{R}^n \times (-T, T)$, prescribing u and u_t on $\{t = 0\}$ lets us solve the equation both forward and backward in time (so with $V = \mathcal{U}$). The hypersurface $\{t = 0\}$ is not the boundary of $\mathcal{U}$, but it can actually be thought as "initial boundary" of (for instance) the future region $\{t > 0\}$.

2.3.2 The non-characteristic condition

To determine a smooth or real analytic solution u, certainly all the derivatives of u must be determined from equations (2.6)–(2.8), and in particular all its derivatives on Σ must be determined by these. Leaving aside the question of constructing the solution, we want to understand why (2.8) are natural to solve the Cauchy problem in the analytic class and what kind of conditions impose. To gain intuition, consider the case $\mathcal{U} = \mathbb{R}^n$ and $\Sigma = \{x_n = 0\}$ a hyperplane, the case of flat Cauchy hypersurface[19]. Then $N = \mathbf{e}_n$ is constant, and (2.8) reads

$$u(x', 0) = g_0(x'), \quad \partial_{x_n} u(x', 0) = g_1(x'), \ \ldots, \ \partial_{x_n}^{k-1} u(x', 0) = g_{k-1}(x'), \qquad x = (x', 0) \in \Sigma.$$

Differentiating (2.8) on Σ along ∂_x^α with $\alpha = (\alpha', j)$, $\alpha' \in \mathbb{N}^{n-1}$, $0 \le j \le k-1$, gives

$$\partial_x^\alpha u(x', 0) = \partial_{x'}^{\alpha'} \partial_{x_n}^j u(x', 0) = \partial_{x'}^{\alpha'} g_j(x'). \tag{2.9}$$

The first missing derivative is $\partial_{x_n}^k u$, which is not prescribed by (2.8). We recover it from the PDE (2.6). Set

$$A(x) := a_{(0,\ldots,0,k)} \left(x, u(x), \nabla u(x), \ldots, \nabla^{k-1} u(x)\right).$$

By (2.9), $A(x)$ is determined on Σ by the Cauchy data $\partial_{x'}^{\alpha'} g_j$ with $|\alpha'| + j \le k-1$. If $A(x) \neq 0$, then on Σ we can single out $\partial_{x_n}^k u(x)$ by rewriting the PDE:

$$\partial_{x_n}^k u(x) = - \sum_{\substack{|\alpha|=k \\ \alpha_n \le k-1}} \frac{a_\alpha(\ldots)}{A(x)} \partial_x^\alpha u(x) \ - \ \frac{a_0(\ldots)}{A(x)}.$$

The right-hand side depends only on the quantities $\partial_{x'}^{\alpha'} g_j$ with $|\alpha'| + j \le k-1$ thanks to (2.9), so $\partial_{x_n}^k u$ is determined on Σ. The condition $A(x) \neq 0$ is precisely the **non-characteristic condition** in the case of flat Cauchy hypersurface.
It allows to actually determine higher-order derivatives. Indeed if we denote $g_k := \partial_{x_n}^k u$ on Σ, which we have just determined from $g_0, \ldots, g_{k-1}$, we can now differentiate the PDE along x_n to obtain

$$\sum_{|\alpha|=k} a_\alpha \left(x, u(x), \nabla u(x), \ldots, \nabla^{k-1} u(x)\right) \partial_x^\alpha \partial_{x_n} u(x) + \tilde{a}_0 \left(x, u(x), \nabla u(x), \ldots, \nabla^k u(x)\right) = 0$$

with the new $\tilde{a}_0$ given by

$$\tilde{a}_0(\ldots) := \sum_{|\alpha|=k} \partial_{x_n} \left[a_\alpha \left(x, u(x), \ldots, \nabla^{k-1} u(x)\right)\right] \partial_x^\alpha u(x) + \partial_{x_n} \left[a_0 \left(x, u(x), \ldots, \nabla^{k-1} u(x)\right)\right],$$

[19] Here and later "flat Cauchy hypersurface" means a hyperplane.

and assuming again $A(x) \neq 0$ on Σ we can compute

$$g_{k+1}(x) := \partial_{x_n}^{k+1} u(x) = -\sum_{|\alpha|=k,\ \alpha_n \leq k-1} \frac{a_\alpha\left(x, \ldots, \nabla^{k-1} u(x)\right)}{A(x)} \partial_x^\alpha \partial_{x_n} u(x) - \frac{\tilde{a}_0\left(x, \ldots, \nabla^k u(x)\right)}{A(x)}$$

so that g_{k+1} is a function of $\partial_{x'}^{\alpha'} g_j$ for $|\alpha'| + j \leq k$ on Σ, and therefore is a function of $\partial_{x'}^{\alpha'} g_j$ for $|\alpha'| + j \leq k-1$, on Σ by the previous step. Taking also derivatives in the tangential directions, this determines $\partial_x^\alpha u$ on Σ for $\alpha = (\alpha', j)$ with $\alpha' \in \mathbb{N}^{n-1}$ and $j \leq k+1$. By induction on k one determines all derivatives of u on Σ. The general condition reads as follow.

Definition 2.18. Given $\Sigma \subset \mathcal{U} \subset \mathbb{R}^n$ a smooth (resp. real analytic) hypersurface as defined above, and $g_0, \ldots, g_{k-1} : \Sigma \to \mathbb{R}$ smooth (resp. real analytic) functions on Σ, we say that the boundary conditions (2.8) are **non-characteristic** for the PDE (2.6) if

$$A(x) := \sum_{|\alpha|=k} a_\alpha\left(x, u(x), \nabla u(x), \ldots, \nabla^{k-1} u(x)\right) N(x)^\alpha \neq 0, \quad \forall x \in \Sigma. \tag{2.10}$$

Note that $A(x)$ only depends on the data (PDE's coefficients a_α with $|\alpha| = k$, the Cauchy hypersurface Σ and the Cauchy data $\{g_j\}_{j=0}^{k-1}$). For the moment, we have justified (2.10) in the flat case; the general case will be derived from this.

It is then natural to ask, given the PDE and the Cauchy data $\{g_j\}_{j=0}^{k-1}$ on which hypersurface we have such condition. This motivates the following definition.

Definition 2.19. Let P be a linear differential operator of order k on $\mathbb{R}^n$, $Pu := \sum_{|\alpha|\leq k} a_\alpha(x)\, \partial_x^\alpha u$, with smooth coefficients $a_\alpha(x)$. The **principal symbol** of P at x is

$$\sigma_P(x, \xi) := \sum_{|\alpha|=k} a_\alpha(x)\, \xi^\alpha, \qquad \xi \in \mathbb{R}^n.$$

Let $\Sigma \subset \mathbb{R}^n$ be a smooth hypersurface and let $x \in \Sigma$. If $N(x) \neq 0$ is a vector normal to Σ at x, we say that Σ is **non-characteristic at** x if $\sigma_P(x, N(x)) \neq 0$. Otherwise we say that Σ is characteristic at x. In terms of the **characteristic cone** at x defined by $C_x := \{\, \xi \in \mathbb{R}^n \setminus \{0\} : \sigma_P(x, \xi) = 0 \,\}$, Σ is characteristic at x if and only if $N(x) \in C_x$. As extension, we call Σ a non-characteristic (resp. characteristic) hypersurface (for P) if it is non-characteristic (resp. characteristic) at every point $x \in \Sigma$.

Remark 2.20 (Quasilinear extension). If P is quasilinear of order k, that is

$$Pu(x) := \sum_{|\alpha|\leq k} a_\alpha\big(x, u(x), \nabla u(x), \ldots, \partial_x^{k-1} u(x)\big)\, \partial_x^\alpha u(x),$$

then for a given function u we *freeze the coefficients at* u and define the *principal symbol along* u by

$$\sigma_P[u](x, \xi) := \sum_{|\alpha|=k} a_\alpha\big(x, u(x), \nabla u(x), \ldots, \partial_x^{k-1} u(x)\big)\, \xi^\alpha, \quad \xi \in \mathbb{R}^n.$$

We then set $C_x[u] := \{\xi \in \mathbb{R}^n \setminus \{0\} : \sigma_P[u](x, \xi) = 0\}$, and say that a hypersurface Σ is non-characteristic at $x \in \Sigma$ (for P along u) if $\sigma_P[u](x, N(x)) \neq 0$, i.e. equivalently $N(x) \notin C_x[u]$. Otherwise Σ is characteristic at x.

Remark 2.21. With these definitions, (2.10) reads exactly $\sigma_P[u](x, N(x)) \neq 0$ on Σ. Hence, the characteristic cone collects, at each point x, the directions that are not admissible, in the sense that the boundary conditions (2.8) are characteristic on any hypersurface with that normal direction. This full directional picture is the natural starting point for the PDE classification (elliptic, hyperbolic, parabolic, etc.).

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Let us justify (2.10). We reduce to the flat case using a local chart Φ. The unknown $v(y) := u(\Psi(y))$, with $\Psi = \Phi^{-1}$, satisfies a k-th order quasilinear PDE

$$\sum_{|\alpha|=k} b_\alpha \left(y, v(y), \nabla v(y), \dots, \nabla^{k-1} v(y)\right) \partial_y^\alpha v + b_0 \left(y, v(y), \nabla v(y), \dots, \nabla^{k-1} v(y)\right) = 0$$

since, from $u(x) = v(\Phi(x))$, by chain rule, ℓ-th order partial derivatives on u depend on $j \leq \ell$-th order derivatives on v. In particular for each multiindex α with $|\alpha| = k$ we write

$$\partial_x^\alpha u(x) = \sum_{|\beta|=k} \widetilde{C}_{\alpha\beta}(x) \left(\partial_y^\beta v\right)(\Phi(x)) \ + \ \text{l.o.t.},$$

where $\widetilde{C}_{\alpha\beta}(x)$ depends on $\nabla_x \Phi(x)$ only, and "l.o.t." collects (lower order) terms involving only $\partial_y^\gamma v$ with $|\gamma| \leq k-1$. For the pure normal multiindex $\beta = (0, \dots, 0, k)$, since on Σ, $\nabla_x \Phi_n = N$,

$$\widetilde{C}_{\alpha,(0,\dots,0,k)}(x) = \prod_{i=1}^n \left(\partial_{x_i} \Phi_n(x)\right)^{\alpha_i} = \left(\nabla_x \Phi_n(x)\right)^\alpha.$$

hence $\widetilde{C}_{\alpha,(0,\dots,0,k)}(x) = N(x)^\alpha$. In the flat case we know that the non-characteristic condition reads $b_{(0,\dots,0,k)}(\dots) \neq 0$. Hence, in the general case we obtain

$$\begin{aligned} 0 \neq b_{(0,\dots,0,k)}(\Phi(x), v(\Phi(x)), \dots, \nabla^{k-1} v(\Phi(x))) &= \sum_{|\alpha|=k} a_\alpha(x, u(x), \dots, \nabla^{k-1} u(x))\, \widetilde{C}_{\alpha,(0,\dots,0,k)}(x) \\ &= \sum_{|\alpha|=k} a_\alpha(x, u(x), \dots, \nabla^{k-1} u(x))\, N(x)^\alpha, \end{aligned}$$

which justifies (2.10). We can now state the key result, saying that (2.10) is not only necessary, but also sufficient to solve the problem locally in the analytic class.

Theorem 2.22 (Cauchy-Kovalevakaya Theorem for PDEs). *Given $\mathcal{U}$, Σ, $g_0, \dots, g_{k-1}$ as above, all real analytic and satisfying the non-characteristic condition at $x \in \Sigma \subset \mathcal{U}$, then there is a unique local analytic solution u to (2.6), (2.8). Namely, there is $\mathcal{U}_x \subset \mathcal{U}$ open set around x so that there is a unique analytic solution u to (2.6) on $\mathcal{U}_x$ satisfying the conditions (2.8) on $\Sigma \cap \mathcal{U}_x$.*

Remark 2.23. This Cauchy-Kovalevskaya theorem was first proved by Cauchy in 1842 for first order quasilinear evolution equations, then formulated in its general form by Kovalevskaya in 1874. At about the same time, Darboux reached similar results, although with less generality than Kovalevskaya. Both Kovalevskaya's and Darboux's papers were published in 1875, and the proof was later simplified by Goursat in his influential calculus textbook around 1900.

2.3.3 Proof of the Cauchy-Kovalevskaya theorem for PDEs

Proof. Step 1. Reduction to a first-order system. First, by analyticity of Σ at $\hat{x}$, we use an analytic chart to reduce to the base point $\hat{x} = 0$ in an open neighborhood $\mathcal{U}_0$ with $x = (\tilde{x}, x_n) \in \mathcal{U}_0$, $\Sigma \cap \mathcal{U}_0 = \{x_n = 0\} \cap \mathcal{U}_0$, and all coefficients a_α still real analytic in $\mathcal{U}_0$. Second, since Σ is non-characteristic at $\hat{x}$, $A(x) := a_{(0,\dots,0,k)}(x) \neq 0$ and after possibly shrinking $\mathcal{U}_0$ we may assume $A(x) \neq 0$ on $\mathcal{U}_0$. Dividing the PDE by $A(x)$ we reduce to $a_{(0,\dots,0,k)}(x) \equiv 1$ on $\mathcal{U}_0$. Thus the PDE can be written in the form

$$\partial_{x_n}^k u(x) = -\sum_{\substack{|\alpha|=k\\ \alpha_n \le k-1}} a_\alpha(x, u(x), \dots, \nabla^{k-1}u(x))\, \partial_x^\alpha u(x) - a_0(x, \dots, \nabla^{k-1}u(x)), \quad \text{for} \quad x \in \mathcal{U}_0. \tag{2.11}$$

Third, we reduce the Cauchy conditions to $\partial_{x_n}^j u(\tilde{x}, 0) = 0$, for $j = 0, \dots, k-1$, by looking at the problem for $v = u - G$ where $G(x) := \sum_{j=0}^{k-1} \frac{x_n^j}{j!} g_j(\tilde{x})$, and then renaming v as u for simplicity. In particular $u(\tilde{x}, 0) = 0$, and hence $u(0) = 0$.
For each multiindex $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ with $|\beta| \le k-1$, define

$$U_\beta(x) := \partial_x^\beta u(x), \qquad U(x) := \big(U_\beta(x)\big)_{|\beta| \le k-1} \in \mathbb{R}^m, \quad m = \binom{n+k-1}{k-1}.$$

We have $\partial_{x_n} U_\beta = \partial_x^{\beta + e_n} u$. Consider three cases. (i) If $|\beta| \le k-2$, then $|\beta + e_n| \le k-1$, so $\partial_{x_n} U_\beta = U_{\beta+e_n}$. (ii) If $|\beta| = k-1$ and $\beta_n \le k-2$, pick $i \le n-1$ with $\beta_i > 0$, set $\alpha := \beta + e_n$ and write $\alpha = e_i + \gamma$ with $|\gamma| = k-1$; then $\partial_{x_n} U_\beta = \partial_x^\alpha u = \partial_{x_i}(\partial_x^\gamma u) = \partial_{x_i} U_\gamma$. (iii) If $\beta = (0, \dots, 0, k-1)$ and $\partial_{x_n} U_\beta = \partial_{x_n}^k u$. In this case we use (2.11), where on the right-hand side each $\partial_x^\alpha u$ is of the form $\partial_{x_i} U_{\gamma'}$ for some γ' with $|\gamma'| = k-1$ as in (ii). Hence, we obtained

$$\partial_{x_n} U_\beta = \sum_{j=1}^{n-1} R_\beta^j(U(x), x)\, \partial_{x_j} U(x) + Q_\beta(U(x), x),$$

for some analytic m row vector R_β^j, and scalar Q_β. Stacking these identities over all $|\beta| \le k-1$, we obtain a first-order quasilinear system

$$\partial_{x_n} U = \sum_{j=1}^{n-1} \widetilde{B}_j(U(x), \tilde{x}, x_n)\, \partial_{x_j} U + \widetilde{B}_0(U(x), \tilde{x}, x_n), \tag{2.12}$$

where $\widetilde{B}_j : \mathbb{R}^m \times \mathbb{R}^{n-1} \times \mathbb{R} \to \mathcal{M}_{m\times m}$ and $\widetilde{B}_0 : \mathbb{R}^m \times \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^m$, with Cauchy data $U(\tilde{x}, 0) = 0$. To remove explicit x_n-dependence in the coefficients, adjoin one more component $U_{m+1}(x) := x_n$, and define

$$u(x) := (U(x), U_{m+1}(x)) \in \mathbb{R}^{\tilde{m}}, \qquad \tilde{m} = m+1.$$

Then (2.12) becomes

$$\partial_{x_n} u = \sum_{j=1}^{n-1} b_j(u(x), \tilde{x})\, \partial_{x_j} u + b_0(u(x), \tilde{x}), \tag{2.13}$$

with Cauchy data $u(\tilde{x}, 0) = 0$, where $b_j : \mathbb{R}^{\tilde{m}} \times \mathbb{R}^{n-1} \to \mathcal{M}_{\tilde{m}\times\tilde{m}}$ for $j = 1, \dots, n-1$ and $b_0 : \mathbb{R}^{\tilde{m}} \times \mathbb{R}^{n-1} \to \mathbb{R}^{\tilde{m}}$ are real analytic and depend only on $(u, \tilde{x})$.

Step 2. Universal polynomials. For a multiindex $\alpha = (\tilde{\alpha}, q) \in \mathbb{N}^{n-1} \times \mathbb{N}$ set $\partial_x^\alpha := \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_{x_n}^q$ and $|\alpha| = |\tilde{\alpha}| + q$. Write $u = (u_1, \dots, u_{\tilde{m}})$. For $j = 0, 1, \dots, n-1$ we regard

$$b_j = b_j(z, \tilde{x}), \qquad (z, \tilde{x}) \in \mathbb{R}^{\tilde{m}} \times \mathbb{R}^{n-1},$$

so in the PDE we evaluate b_j at $(z, \tilde{x}) = (u(x), \tilde{x})$. Let $b_{j,\ell}$ be the ℓ-th component of b_j, and define

$$B_{j,\ell,\beta} := \partial_{(z,\tilde{x})}^\beta b_{j,\ell}(0,0),$$

for any multiindex β in the $(z, \tilde{x})$-variables. We claim that for each $\alpha = (\tilde{\alpha}, q)$ with $q \geq 0$ and each i, there exists a polynomial $P_{\alpha,i}$ with nonnegative integer coefficients such that

$$\partial_x^\alpha u_i(0) = P_{\alpha,i}\big(B_{j,\ell,\beta}\big), \tag{2.14}$$

where the arguments $B_{j,\ell,\beta}$ range over all $j = 0, \dots, n-1$, $\ell = 1, \dots, \tilde{m}$, and all β with $|\beta| \leq |\alpha| - 1$. We proceed by induction on q. From the Cauchy data we have $u(\tilde{x}, 0) = 0$, hence $\partial_{\tilde{x}}^{\tilde{\alpha}} u(\tilde{x}, 0)|_{\tilde{x}=0} = 0$ for all $\tilde{\alpha}$. Thus for $\alpha = (\tilde{\alpha}, 0)$, $\partial_x^\alpha u_i(0) = 0$, which is (2.14) with $P_{\alpha,i} \equiv 0$. Assume (2.14) holds for all $(\tilde{\gamma}, q')$ with $q' < q$, and fix $\alpha = (\tilde{\alpha}, q)$ with $q \geq 1$. Apply $\partial_{\tilde{x}}^{\tilde{\alpha}} \partial_{x_n}^{q-1}$ to (2.13) then evaluate at $x = 0$. On the left-hand side we have $\partial_x^\alpha u_i(0)$. On the right-hand side, for $j \geq 1$, Leibniz' rule gives a sum of terms

$$\tilde{C}_{\tilde{\mu},r} \left(\partial_{\tilde{x}}^{\tilde{\mu}} \partial_{x_n}^r [\, b_j(u, \tilde{x}) \,]|_0\right) \left(\partial_{\tilde{x}}^{\tilde{\alpha}-\tilde{\mu}} \partial_{x_n}^{q-1-r} [\, \partial_{x_j} u \,]|_0\right), \qquad \tilde{\mu} \leq \tilde{\alpha},\ 0 \leq r \leq q-1,$$

where $\tilde{C}_{\tilde{\mu},r}$ are integer positive coefficients (deriving from product rule with multiindices). Each factor of the first type is computed by the chain rule on the composition $x \mapsto (u(x), \tilde{x}) \mapsto b_j(z, \tilde{x})$. This is a finite sum of terms of the form $\hat{C} B_{j,\ell,\beta} \prod_s \partial_x^{\gamma^{(s)}} u_{i_s}(0)$, where $\hat{C}$ is a positive integer (deriving from the iterated chain rule), β is a multiindex in the $(z, \tilde{x})$-variables, and the factors $\partial_x^{\gamma^{(s)}} u_{i_s}(0)$ arise from the z-derivatives encoded in β. In particular, each $\gamma^{(s)} = (\tilde{\gamma}^{(s)}, q_s)$ satisfies $q_s \leq r \leq q-1$, so we can invoke the induction hypothesis to write them in terms of polynomials of $B_{j,\ell,\beta}$. For the second factor, $\partial_{\tilde{x}}^{\tilde{\alpha}-\tilde{\mu}} \partial_{x_n}^{q-1-r} [\, \partial_{x_j} u \,]|_0 = \partial_x^\delta u(0)$ for some $\delta = (\tilde{\delta}, q')$ with $q' = q - 1 - r \leq q - 1$. Since $q' < q$ and we again invoke the induction hypothesis. Thus this second factor is also a polynomial in the admissible $B_{j,\ell,\beta}$. The b_0-term is handled in the same way (without Leibniz' rule). Therefore $\partial_x^\alpha u_i(0)$ is a polynomial in the $B_{j,\ell,\beta}$ with $|\beta| \leq |\alpha| - 1$, with nonnegative integer coefficients, proving (2.14) for level q.

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Step 3. Convergence of the candidate solution. Renaming $\tilde{m}$ to m for simplicity, we define

$$u_i(x) := \sum_{\alpha\in\mathbb{N}^n} \frac{x^\alpha}{\alpha!}\,\partial_x^\alpha u_i(0), \qquad i = 1,\dots,m, \tag{2.15}$$

where, by the previous step, the coefficients $\partial_x^\alpha u_i(0)$ are determined by $P_{i,\alpha}$ computed at $B_{j,\ell,\beta}$, hence are uniquely determined by the PDE and the Cauchy data. To prove that the series (2.15) converges, we construct a majorant system. Let $C, r > 0$ and define

$$g(x,z) := \frac{Cr}{r - \sum_{j=1}^{n-1} x_j - \sum_{k=1}^m z_k} = C\sum_{q\geq 0}\left(\tfrac{\sum_{j=1}^{n-1} x_j + \sum_{k=1}^m z_k}{r}\right)^q,$$

which is analytic if $\sum_{j=1}^{n-1}|x_j| + \sum_{k=1}^m |z_k| < r$. Since b_j, b_0 are real analytic near 0, the Taylor coefficients of their components are bounded in absolute value by those of g, if we choose $C > 0$ large and $r > 0$ small enough. Hence, defining

$$b_j^* := g\,\mathbf{M}_1, \qquad b_0^* := g\,\mathbf{U}_1,$$

where $\mathbf{M}_1$ is the $m\times m$ matrix with 1 in all entries and $\mathbf{U}_1$ is the m-vector with 1 in all entries. We obtain $b_j^* \gg b_j$ and $b_0^* \gg b_0$. Consider the auxiliary Cauchy problem

$$\partial_{x_n} v = \sum_{j=1}^{n-1} b_j^*(v,\tilde{x})\,\partial_{x_j} v + b_0^*(v,\tilde{x}), \qquad \text{with} \quad v(\tilde{x},0) = 0 \quad \text{on} \quad \Sigma. \tag{2.16}$$

The solution is given by (verify it)

$$v(x) = \frac{(r - \sum_{j=1}^{n-1} x_j) - \sqrt{\left(r - \sum_{j=1}^{n-1} x_j\right)^2 - 2nmCr\,x_n}}{nm}\,\mathbf{U}_1 \tag{2.17}$$

that is real analytic in all variables near zero. This concludes the proof, since we have, defining $B_{j,\ell,\beta}^* = \partial_{(z,\tilde{x})}^\beta b_{j,\ell}^*(0,0)$,

$$|\partial_x^\alpha u_i(0)| = |P_{\alpha,i}\left(B_{j,\ell,\beta}\right)| \leq P_{\alpha,i}\left(|B_{j,\ell,\beta}|\right) \leq P_{\alpha,i}(B_{j,\ell,\beta}^*) = \partial_x^\alpha v_i(0),$$

so that $v \gg u$ at zero and the Taylor series (2.15) has a non-zero radius of analyticity. Uniqueness follows because any other analytic solution satisfies the same recursion (2.14), hence has the same Taylor coefficients at 0 and must agree with u in a neighborhood of $\hat{x}$. ☐

Remark 2.24 (Solving (2.16)). By symmetry we look for $v(x) = w(\xi,t)\,\mathbf{U}_1$, where $\mathbf{U}_1 = (1,\dots,1) \in \mathbb{R}^m$, $\xi := x_1 + \dots + x_{n-1}$, $t := x_n$. Then $v = 0$ on $\{t = 0\}$ becomes $w(\xi,0) = 0$. Since all components of v are equal, the coefficients depend on v only through $\sum_{\alpha=1}^m v_\alpha = mw$, and on $\tilde{x}$ only through ξ, so $b_j^*(v,\tilde{x}) = \frac{mCr}{r-\xi-mw}\mathbf{U}_1$ for $j \leq n-1$ and $b_0^*(v,\tilde{x}) = \frac{Cr}{r-\xi-mw}\mathbf{U}_1$. Using $\partial_{x_j} v = \partial_\xi w \mathbf{U}_1$ for $j \leq n-1$ and $\partial_{x_n} v = \partial_t w \mathbf{U}_1$, the system reduces to

$$\partial_t w = \frac{Cr}{r - \xi - mw}\Big[(n-1)m\,\partial_\xi w + 1\Big], \qquad w(\xi,0) = 0.$$

Write this as $\partial_t w - a\,\partial_\xi w = b$ with $a = \frac{(n-1)mCr}{r-\xi-mw}$, $b = \frac{Cr}{r-\xi-mw}$, and solve it by *the method of characteristics* (that is the same method applied for solving Exercise 1.9, but here we need to deal with a and b depending on the solution), to find

$$v(x) = \frac{(r - \sum_{j=1}^{n-1} x_j) - \sqrt{(r - \sum_{j=1}^{n-1} x_j)^2 - 2nmCr\,x_n}}{nm}\,\mathbf{U}_1,$$

which solves the PDE with $v = 0$ on $\{x_n = 0\}$, and it is analytic near zero.

2.4 Limitations and classification

2.4.1 Limitations of the Cauchy-Kovalevskaya theorem

We list some limitations while looking for analytic solutions and using Cauchy-Kovalevskaya theorem.

- Exercise 1.10 studies Kovalevskaya's counterexample for the heat equation $\partial_t u = \partial_x^2 u$ on $\mathbb{R}^2$ with analytic data $u(0,x) = \frac{1}{1+x^2}$. With $\Sigma = \{t = 0\}$, the hypersurface is characteristic: the condition $a_{2,0} \neq 0$ never holds, independently of the boundary data. Thus, Cauchy-Kovalevskaya theorem does not apply. However, it can be seen with other tools (e.g. heat semigroup, analytic energy methods) that the forward Cauchy problem (for $t > 0$) for the heat equation is well posed[20] (for example in the classical spaces C^k with $k \geq 2$) but the backward ($t < 0$) Cauchy problem is ill-posed. This shows a first limitation: *the theorem studies the problem in both forward and backward directions at once.*
- Transport and wave equations have *localized effects* such as "finite speed of propagation": a local perturbation on the hypersurface as an influence in a finite spacetime region. However, analytic functions are globally determined from their local behaviour, so the analytic framework cannot capture such qualitative properties. This is a limitation of working in the analytic class.
- Another limitation of working inside the class of smooth solutions is that we cannot understand the *regularisation effect* of the equation at hand. For instance any C^2 solution to $\Delta u = 0$ is automatically smooth, and in fact real analytic. This promotion happens thanks to the special structure of the equation, and it doesn't happen in general.
- With Cauchy–Kovalevskaya theorem *we cannot exclude the existence of other non analytic solutions.* Uniqueness in the C^∞ class can fail for non-characteristic Cauchy problems (except in the case of *linear* k-th order PDEs where Holmgren's theorem shows that C^k solutions to non-characteristic PDEs with real analytic coefficients are real analytic near Σ).
- *There is no general local existence result if we drop analyticity assumptions on data or coefficients.*
  - *Non-analytic Cauchy data.* For the (full) Laplace equation $\Delta_{x,t} u = 0$ in variables (x,t) with $\Sigma = \{t = 0\}$ which is non-characteristic[21], every harmonic solution is real-analytic; hence its traces $u|_\Sigma$ and $\partial_\nu u|_\Sigma$ are real-analytic. Therefore there is no solution if we prescribe C^∞ but not analytic data on Σ.
  - *Non-analytic coefficients.* Even with smooth coefficients, local solvability can fail: Lewy's 1957 counterexample gives a smooth linear partial differential equation without solution.
- The Cauchy problem for elliptic PDEs is intrinsically ill-posed in classical spaces such as C^k, and the analyticity in the Cauchy-Kovalevskaya theorem is hiding this, as Hadamard's example shows (see Exercise 2.1). For $(\partial_t^2 + \partial_x^2)u = 0$ on $\mathbb{R}^2$ with $u(x,0) = 0$ and $\partial_t u(x,0) = \cos(\omega x)$, the solution is $u(x,t) = \frac{1}{\omega}\sinh(\omega t)\cos(\omega x)$. Thus, for $\omega \gg 1$, the data are $\mathcal{O}(1)$ while $u(x,1) = \mathcal{O}(e^\omega/\omega)$ in L^∞, so the solution operator is unbounded as $\omega \to \infty$. Thus, despite applicability of Cauchy-Kovalevskaya theorem, the Cauchy

[20] For the heat equation with space domain $\mathbb{R}^n$ we need some restriction on the growth of $u(x,t)$ as $|x| \to \infty$ to guarantee uniqueness.

[21] This is different from $\Delta_{x,y} u = 0$ with t as a third variable, where $\{t = 0\}$ is characteristic and the Cauchy–Kovalevskaya theorem does not apply.

problem is not the right framework for elliptic equations. In Chapter 4 we will see that the boundary value problem is the correct framework.

2.4.2 Classification of PDEs

The Laplace $\Delta u = 0$ and Poisson $\Delta u = f$ equations have $\sigma_p(x,\xi) = |\xi|^2$ and the cone[22] $\mathcal{C}_x = \{0\}$ for any $x \in \mathbb{R}^n$. Equations without characteristic hypersurfaces are called **elliptic equations**. Trying to capture the essence of the poor behaviour of the Laplace and Cauchy-Riemann equations in relation to their Cauchy initial-time problems leads to the concept of *ellipticity*. Ellipticity means that matrix (a_{ij}) has all eigenvalues strictly positive or all strictly negative.

The wave equation $\Box u = -\partial^2_{x_n} u + \sum_{j=1}^{n-1} \partial^2_{x_j} u = 0$ has $\sigma_p(x,\xi) = \xi_1^2 + \cdots + \xi_{n-1}^2 - \xi_n^2$, and $\mathcal{C}_x = \{\xi_n^2 = \xi_1^2 + \cdots + \xi_{n-1}^2\}$, the (sound/light) cone. Hypersurfaces whose normal makes an angle $\pi/4$ with the direction e_n are characteristic; here $x_n = t$ represents time. The transport equation $\sum_{j=1}^n c_j(x)\,\partial_{x_j} u = 0$ has $\sigma_p(x,\xi) = \vec{c}(x)\cdot\xi$, and $\mathcal{C}_x = \vec{c}(x)^\perp$. Hypersurfaces tangent to $\vec{c}$ are characteristic. These are examples of **hyperbolic equations**. The idea of *hyperbolicity* is an attempt to identify the class of PDEs for which the Cauchy-Kovalevskaya theorem can be rescued in some sense when we relax the analyticity assumption.

The heat equation $\partial_t u = \Delta_x u$ has principal symbol $\sum_{i=1}^{n-1} \xi_i^2$ and $\mathcal{C}_x = \{\xi_1 = \cdots = x_{n-1} = 0\}$, so the characteristic hypersurfaces are time slices $\{t = \text{const}\}$; such equations are *parabolic*. The class of **parabolic equations** is a class for which the evolution problem is well-posed for positive times, but is ill-posed for negative times. The initial condition is characteristic and the Cauchy-Kovalevskaya theorem fails. The informations is transmitted at infinite speed, and the solution becomes analytic for positive times (regularisation).

The Schrödinger equation $i\partial_t u + \Delta u = 0$ has the same principal symbol and is *dispersive*. The class of **dispersive equations** is a class in between hyperbolic equations (local well-posedness for both forward and backward times, finite speed of propagation) and parabolic equations (initial conditions are characteristic). To see why it's called "dispersive," take the same 1-frequency ansatz $u(t,x) = e^{i(kx-\omega t)}$. Plugging into the *wave equation* $u_{tt} - u_{xx} = 0$ gives $-\omega^2 + k^2 = 0$, so $\omega = |k|$, instead into *Schrödinger* $i\,u_t + u_{xx} = 0$ gives $\omega = k^2$. The map $\omega = \omega(k)$ is the dispersion relation. Here we are interested in the region where $|u|^2$ is concentrated (a localized bump). Its speed is read from how ω changes with k. Thus, for the wave equation $\mathrm{d}\omega/\mathrm{d}k = \pm 1$ (no spreading), while for Schrödinger $\mathrm{d}\omega/\mathrm{d}k = 2k$ (depends on k): different frequencies move at different speeds, so a bump is dispersing.

There are also of course **equations of mixed type**, e.g. the Euler-Tricomi equation $\partial^2_{xx} u = x\partial^2_{yy} u$ in $\mathbb{R}^2$ which is hyperbolic in the region $\{x > 0\}$ and elliptic in the region $\{x < 0\}$. There are also variants of these classes where some properties are weakened, e.g. the **hypoelliptic equations** pioneered by Kolmogorov and Hörmander.

[22] The principal symbol is a k-homogeneous function in ξ, which explains the name "cone" for $\mathcal{C}_x$.

LECTURE 10

3 Functional toolbox

We review Hölder and Lebesgue spaces, then introduce weak derivatives and Sobolev spaces, which measure regularity in an integral rather than pointwise sense. The aims are:

(i) to use Banach and Hilbertian techniques;

(ii) to work in spaces tracking energies or other physical quantities that are minimized in elliptic PDEs and propagated in hyperbolic PDEs.

We then study approximation in Sobolev spaces, extension and trace results, and Sobolev inequalities, which trade integrability of derivatives for improved regularity, integral or even pointwise, of the function. We conclude with compactness results for Sobolev spaces.

3.1 Hölder Spaces

Definition 3.1 (Classical C^k spaces)**.** Let $\mathcal{U} \subset \mathbb{R}^n$ be open and $k \in \mathbb{N}_0 \cup \{\infty\}$.

$$\begin{aligned}
C^k(\mathcal{U}) &:= \{\, u : \mathcal{U} \to \mathbb{R} : \partial_x^\alpha u \text{ exists and is continuous on } \mathcal{U}\ \forall\, |\alpha| \le k \,\},\\
C_b^k(\mathcal{U}) &:= \{\, u \in C^k(\mathcal{U}) : \sup_{x\in\mathcal{U}} |\partial_x^\alpha u(x)| < \infty\ \forall\, |\alpha| \le k \,\},\\
C^k(\overline{\mathcal{U}}) &:= \{\, u \in C_b^k(\mathcal{U}) : \partial_x^\alpha u \text{ is uniformly continuous on } \mathcal{U}\ \forall\, |\alpha| \le k \,\}.
\end{aligned}$$

Remark 3.2. Let $k < \infty$. $C^k(\mathcal{U})$ does not admit a distance induce by a norm.[23] In contrast, $C_b^k(\mathcal{U})$ and $C^k(\overline{\mathcal{U}})$ are Banach spaces with the norm

$$\|u\|_{C^k} := \max_{|\alpha|\le k} \sup_{x\in\mathcal{U}} |\partial^\alpha u(x)|.$$

Remark 3.3. Note that $C^k(\overline{\mathcal{U}})$ is defined via the behaviour on the *open* set $\mathcal{U}$, not by properties on $\overline{\mathcal{U}}$. Accordingly, the set of functions that are C^k on $\overline{\mathcal{U}}$ *is different*, on unbounded sets, from our definition of $C^k(\overline{\mathcal{U}})$. In particular, $C^k(\overline{\mathbb{R}^n}) \neq C^k(\mathbb{R}^n)$ under our convention (check it), even though $\overline{\mathbb{R}^n} = \mathbb{R}^n$.

We turn to Hölder spaces that interpolate in between the C^k spaces.

Definition 3.4. Let $\mathcal{U} \subset \mathbb{R}^n$ be open and $\gamma \in (0,1]$. Define the seminorm

$$[u]_{C^{0,\gamma}(\overline{\mathcal{U}})} := \sup_{\substack{x,y\in\mathcal{U}\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

The 0-*Hölder space with index* γ is

$$C^{0,\gamma}(\overline{\mathcal{U}}) := \Big\{\, u : \mathcal{U} \to \mathbb{R} \text{ bounded} \;\Big|\; [u]_{C^{0,\gamma}(\overline{\mathcal{U}})} < \infty \Big\}.$$

For $k \in \mathbb{N}$, the k-*Hölder space with index* γ is

$$C^{k,\gamma}(\overline{\mathcal{U}}) := \Big\{\, u \in C^k(\overline{\mathcal{U}}) \;\Big|\; \partial_x^\alpha u \in C^{0,\gamma}(\overline{\mathcal{U}}) \text{ for all } |\alpha| \le k \Big\}.$$

[23] With the C^k natural topology, that is $u_j \to u$ if for every compact $K \Subset \mathcal{U}$ and every $|\alpha| \le k$, $\sup_{x\in K} |\partial^\alpha (u_j - u)(x)| \to 0$, $C^k(\mathcal{U})$ is not normable. Instead it is a Fréchet space: complete and metrizable, with topology determined by a countable family of seminorms rather than a single norm.

These spaces are Banach spaces for the norm (check it)

$$\|u\|_{C^{k,\gamma}(\overline{\mathcal{U}})} := \|u\|_{C^k(\overline{\mathcal{U}})} + \sum_{|\alpha|\le k} \left[\partial_x^\alpha u\right]_{C^{0,\gamma}(\overline{\mathcal{U}})}.$$

Remark 3.5. $C^{0,1}(\overline{\mathcal{U}})$ is the space of Lipschitz functions.
Taking $\gamma > 1$ in the Hölder condition forces u to be differentiable with $\nabla u = 0$, hence u is constant.
Hölder continuity on $\mathcal{U}$ with a uniform constant $C > 0$ (as in our definition) implies uniform continuity, which justifies writing these spaces with $\overline{\mathcal{U}}$ rather than $\mathcal{U}$.
A norm equivalent to $\|u\|_{C^{k,\gamma}(\overline{\mathcal{U}})}$ is

$$\|u\|'_{C^{k,\gamma}(\overline{\mathcal{U}})} = \|u\|_{C^k(\overline{\mathcal{U}})} + \sum_{|\alpha|=k} \left[\partial_x^\alpha u\right]_{C^{0,\gamma}(\overline{\mathcal{U}})}.$$

3.2 Lebesgue spaces

Definition 3.6 (Lebesgue spaces)**.** Let $\mathcal{U} \subset \mathbb{R}^n$ be open and $p \in [1,\infty]$. The *Lebesgue space* $L^p(\mathcal{U})$ consists of measurable $u : \mathcal{U} \to \mathbb{R}$, considered as equivalence classes up to equality almost everywhere (a.e.), such that $\int_{\mathcal{U}} |u|^p \, dx < \infty$ if $p \in [1,\infty)$ or $\operatorname{ess\,sup}_{\mathcal{U}} |u| < \infty$ if $p = \infty$. The *local Lebesgue space* is

$$L^p_{\mathrm{loc}}(\mathcal{U}) := \{u : \mathcal{U} \to \mathbb{R} \text{ measurable on } \mathcal{U} : \ u \in L^p(\mathcal{V}) \text{ for every open } \mathcal{V} \Subset \mathcal{U}\},$$

where $\mathcal{V} \Subset \mathcal{U}$ means that the closure $\overline{\mathcal{V}}$ is compact and included in $\mathcal{U}$.

We define the associated norms

$$\|u\|_{L^p(\mathcal{U})} := \begin{cases} \left(\int_{\mathcal{U}} |u|^p \, dx \right)^{1/p}, & \text{for } \ 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathcal{U}} |u(x)|, & \text{for } \ p = \infty. \end{cases}$$

$L^p(\mathcal{U})$ is a Banach space.[24] For $p = 2$, with $\langle u, v\rangle = \int_{\mathcal{U}} uv \, dx$, $L^2(\mathcal{U})$ is a Hilbert space.

The main theorems of Lebesgue integration theory are the following. Assume $\{f_n\}_n$ are measurable functions.

1. *Monotone convergence:* If $0 \le f_1 \le f_2 \le \cdots$ and $f_n \uparrow f$ a.e., then $\int_{\mathcal{U}} f_n \, dx \uparrow \int_{\mathcal{U}} f \, dx$.
2. *Fatou's lemma:* For nonnegative $\{f_n\}$, $\int_{\mathcal{U}} \liminf_n f_n \, dx \le \liminf_n \int_{\mathcal{U}} f_n \, dx$.
3. *Dominated convergence:* If $f_n \to f$ a.e. and $|f_n| \le g$ a.e. with $g \in L^1(\mathcal{U})$, then $\int_{\mathcal{U}} f_n \, dx \to \int_{\mathcal{U}} f \, dx$.

3.3 Weak (generalised) derivatives

In order to measure regularity through integrals and define Sobolev spaces, it is natural to introduce a generalised notion of differentiability.

Definition 3.7. Given $\mathcal{U} \subset \mathbb{R}^n$ open, $u, v \in L^1_{loc}(\mathcal{U})$, and $\alpha \in \mathbb{N}^n$, we say that v is the α *weak derivative* of u, denoted by $v = D_x^\alpha u$, if

$$\forall \varphi \in C_c^\infty(\mathcal{U}), \qquad \int_{\mathcal{U}} u \partial_x^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_{\mathcal{U}} v\varphi \, \mathrm{d}x.$$

[24] $L^p_{\mathrm{loc}}(\mathcal{U})$ is not normed but is Fréchet.

This means that the weak derivative verifies the formula of integration by parts, provided we avoid boundaries (using compact supported test functions).

Remark 3.8 (Extension to distributions). A *distribution* on an open set $\mathcal{U} \subset \mathbb{R}^n$ is a continuous linear functional $T : \mathcal{C}_c^\infty(\mathcal{U}) \to \mathbb{R}$; the space of distributions is $\mathcal{D}'(\mathcal{U})$. For any multi-index α, the *distributional derivative* $\partial^\alpha T \in \mathcal{D}'(\mathcal{U})$ is defined by $\langle \partial^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle$ for $\varphi \in \mathcal{C}_c^\infty(\mathcal{U})$. This extends the weak derivative definition for $L^1_{\mathrm{loc}}(\mathcal{U})$ functions to $\mathcal{D}'(\mathcal{U})$. As an example, let $H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x > 0$. Then $D_x H = \delta_0$, the Dirac Delta, in $\mathcal{D}'(\mathbb{R})$; hence H has a distributional derivative but no L^1_{loc} weak derivative, since $\delta_0 \notin L^1_{\mathrm{loc}}(\mathbb{R})$.

Remark 3.9. The weak derivative, when it exists in $L^1_{\mathrm{loc}}(\mathcal{U})$, is unique. This follows by the *fundamental lemma of the calculus of variations*, saying that if $w \in L^1_{\mathrm{loc}}(\mathcal{U})$ and $\int_{\mathcal{U}} w\varphi \, dx = 0$ for all $\varphi \in C_c^\infty(\mathcal{U})$, then $w = 0$ a.e. in $\mathcal{U}$. In particular, if $u \in C^k(\mathcal{U})$ and $|\alpha| \leq k$, then the weak and classical derivatives coincide (prove it).

LECTURE 11

Definition 3.10. Let $\mathcal{U} \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(\mathcal{U})$ is

$$W^{k,p}(\mathcal{U}) := \left\{ u \in L^1_{\text{loc}}(\mathcal{U}) : D^\alpha u \text{ exists and } D^\alpha u \in L^p(\mathcal{U}) \text{ for all } |\alpha| \le k \right\}.$$

It is equipped with the norm

$$\|u\|_{W^{k,p}(\mathcal{U})} = \begin{cases} \Big(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\mathcal{U})} \Big)^{1/p}, & 1 \le p < \infty, \\ \max_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty(\mathcal{U})}, & p = \infty. \end{cases}$$

We also define the subspace $W^{k,p}_0(\mathcal{U})$ as $W^{k,p}_0(\mathcal{U}) := \overline{C^\infty_c(\mathcal{U})}^{\|\cdot\|_{W^{k,p}(\mathcal{U})}}$, that is the closure of $C^\infty_c(\mathcal{U})$ in $W^{k,p}(\mathcal{U})$.

Remark 3.11. $(W^{k,p}(\mathcal{U}), \|\cdot\|_{W^{k,p}(\mathcal{U})})$ is a Banach space. The completeness of $(W^{k,p}(\mathcal{U}), \|\cdot\|_{W^{k,p}})$ follows from the completeness of $L^p(\mathcal{U})$ and the fact that if $u_j \to u$ and $D^\alpha u_j \to v_\alpha$ in $L^p(\mathcal{U})$ for all $|\alpha| \le k$, then $v_\alpha = D^\alpha u$. For $p = 2$ we write $H^k(\mathcal{U}) := W^{k,2}(\mathcal{U})$ and $H^k_0(\mathcal{U}) := W^{k,2}_0(\mathcal{U})$; these are Hilbert spaces with inner product $\langle u, v \rangle_{H^k(\mathcal{U})} := \sum_{|\alpha| \le k} \int_{\mathcal{U}} D^\alpha u(x)\, D^\alpha v(x), dx$.

Example 3.12. Let $u(x) = |x|^{-s}$ on $B(0,1)$. One checks that $u \in L^1(B(0,1))$ if and only if $s < n$, and $u \in W^{1,p}(B(0,1))$ if and only if $s < \frac{n-p}{p}$, with $D_{x_i} u(x) = -s\, x_i\, |x|^{-s-2}$ (the weak derivative) in $B(0,1)$. In particular, if we require $u \in W^{1,p}(B(0,1))$ with $p > n$, then the restriction $s < \frac{n-p}{p}$ shows that u is actually continuous; this is reminiscent of the Sobolev(-Morrey) inequalities that we will discuss later.

3.4 Approximation in Sobolev spaces

From our definition of $W^{k,p}$ via weak derivatives, it is not obvious that Sobolev functions can be approximated by regular ones: Meyers–Serrin theorem[25] (which is Proposition 3.14 (5)) shows that C^∞ is dense in $W^{k,p}$. This lets us prove statements for smooth functions and then take limits to get the analogue statement for Sobolev functions.

Definition 3.13. A family $(\phi_\varepsilon)_{\varepsilon>0} \subset C^\infty_c(\mathbb{R}^n)$ is a *standard mollifier* if, for every $\varepsilon > 0$,

$$\operatorname{supp} \phi_\varepsilon \subset \overline{B(0,\varepsilon)}, \qquad \phi_\varepsilon \ge 0, \qquad \int_{\mathbb{R}^n} \phi_\varepsilon(x)\, dx = 1.$$

If $\mathcal{U} \subset \mathbb{R}^n$ is open and $u \in L^1_{\text{loc}}(\mathcal{U})$, the *mollification* $u_\varepsilon : \mathcal{U}_\varepsilon \to \mathbb{R}$ of u at scale ε is

$$u_\varepsilon := \phi_\varepsilon * u \quad \text{on} \quad \mathcal{U}_\varepsilon := \{x \in \mathcal{U} : \operatorname{dist}(x, \partial\mathcal{U}) > \varepsilon\}.$$

Proposition 3.14. *Let $\mathcal{U} \subset \mathbb{R}^n$ open, $k \in \mathbb{N}$ and $p \in [1, +\infty)$.*

1. *There exists a standard mollifier.*
2. *If $u \in L^1_{loc}(\mathcal{U})$, the mollification $u_\varepsilon \in C^\infty(\mathcal{U}_\varepsilon)$ with $u_\varepsilon \to u$ in $L^1_{loc}(\mathcal{U})$, i.e. in $L^1(\mathcal{V})$ for any $\mathcal{V} \subset\subset \mathcal{U}$, and almost everywhere in $\mathcal{U}$.*

[25] Historically, Sobolev spaces were defined either via weak derivatives or as the closure of smooth functions; the Meyers–Serrin theorem shows these definitions coincide. See Meyers–Serrin "H = W" (1964).

3. *If $u \in C^k(\mathcal{U})$, then, for $|\alpha| \le k$, $\partial_x^\alpha u_\varepsilon \to \partial_x^\alpha u$ uniformly on compact subsets of $\mathcal{U}$.*
4. *(Local Sobolev smoothing) If $u \in W^{k,p}(\mathcal{U})$, then $u_\varepsilon \to u$ in $W^{k,p}_{loc}(\mathcal{U})$, i.e. in $W^{k,p}(\mathcal{V})$ for any open $\mathcal{V} \subset\subset \mathcal{U}$.*
5. *(Global approximation away from the boundary) If $\mathcal{U}$ is bounded and $u \in W^{k,p}(\mathcal{U})$, then there exists a sequence $u_j \in C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U})$ so that $u_j \to u$ in $W^{k,p}(\mathcal{U})$.*
6. *(Global approximation up to $\partial\mathcal{U}$) Given $\mathcal{U}$ bounded with $\partial\mathcal{U}$ being locally the graph of a Lipschitz function, and $u \in W^{k,p}(\mathcal{U})$, there is a sequence $u_j \in C^\infty(\overline{\mathcal{U}})$ such that $u_j \to u$ in $W^{k,p}(\mathcal{U})$.*

Proof of Proposition 3.14. 1. Use for instance $\varphi(x) = C\exp(-(1-|x|^2)^{-1})$ on $B(0,1)$ and $\varphi \equiv 0$ outside $B(0,1)$ with a well-chosen $C > 0$, and rescale it $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

2. Using the definitions it is easy to see that u_ε is continuous, and then differentiable in any coordinate direction $\{e_j\}_{j=1}^n$ by taking the limit as $h \to 0$ of the difference quotient $h^{-1}[u_\varepsilon(x+he_j) - u_\varepsilon(x)]$. Then iterate the argument and deduce $\partial^\alpha u_\varepsilon = (\partial^\alpha \rho_\varepsilon) * u \in C^\infty(\mathcal{U}_\varepsilon)$.

Fix $\mathcal{V} \Subset \mathcal{U}$ and set $\varepsilon_0 := \frac{1}{2}\operatorname{dist}(\mathcal{V}, \partial\mathcal{U}) > 0$. For $0 < \varepsilon \le \varepsilon_0$ and $x \in \mathcal{V}$ we have

$$u_\varepsilon(x) - u(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\big(u(x-y) - u(x)\big)\,dy,$$

hence, by Fubini,

$$\|u_\varepsilon - u\|_{L^1(\mathcal{V})} \le \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\,\|\tau_{-y}u - u\|_{L^1(\mathcal{V})}\,dy, \qquad \text{where} \quad \tau_h u(x) := u(x+h). \tag{3.1}$$

We recall that $\|\tau_h u - u\|_{L^1(\mathcal{V})} \to 0$ as $h \to 0$ (continuity of translations in L^1_{loc}). Indeed, choose $\mathcal{W}$ with $\mathcal{V} \Subset \mathcal{W} \Subset \mathcal{U}$ and $\phi \in C_c^\infty(\mathcal{W})$ with $\|u - \phi\|_{L^1(\mathcal{W})} < \varepsilon$ (density of smooths in L^1). For $|h|$ small so that $\mathcal{V} + h \subset \mathcal{W}$,

$$\|\tau_h u - u\|_{L^1(\mathcal{V})} \le \|\tau_h(u-\phi)\|_{L^1(\mathcal{V})} + \|\tau_h\phi - \phi\|_{L^1(\mathcal{V})} + \|\phi - u\|_{L^1(\mathcal{V})} < 2\varepsilon + \|\tau_h\phi - \phi\|_{L^1(\mathcal{V})}.$$

Since ϕ is smooth with compact support, $\tau_h\phi \to \phi$ uniformly, hence in $L^1(\mathcal{V})$; thus the last term is $< \varepsilon$ for $|h|$ small. Therefore $\|\tau_h u - u\|_{L^1(\mathcal{V})} \to 0$, and because $\operatorname{supp}\varphi_\varepsilon \subset B(0,\varepsilon)$ and $\int \varphi_\varepsilon = 1$, the right-hand side of (3.1) tends to 0 as $\varepsilon \to 0$. This proves $u_\varepsilon \to u$ in $L^1_{\text{loc}}(\mathcal{U})$.

For almost everywhere convergence, note that $\varphi_\varepsilon \ge 0$, $\int \varphi_\varepsilon = 1$, and $\operatorname{supp}\varphi_\varepsilon \subset B(0,\varepsilon)$ give

$$|u_\varepsilon(x) - u(x)| \le \int \varphi_\varepsilon(y)\,|u(x-y) - u(x)|\,dy \le \frac{\|\varphi\|_\infty}{|B(0,\varepsilon)|}\int_{B(0,\varepsilon)} |u(x-y) - u(x)|\,dy.$$

By the Lebesgue differentiation theorem (the averages of u over smaller and smaller balls tends to $u(x)$ for almost every point x, the Lebesgue points), the right-hand side $\to 0$ at every Lebesgue point of u. Hence $u_\varepsilon(x) \to u(x)$ for a.e. $x \in \mathcal{U}$.

3. Let $u \in C^k(\mathcal{U})$ and fix $\mathcal{V} \Subset \mathcal{U}$. Set $\varepsilon_0 := \frac{1}{2}\operatorname{dist}(\mathcal{V}, \partial\mathcal{U}) > 0$, so $\mathcal{V} \subset \mathcal{U}_\varepsilon$ for $0 < \varepsilon \le \varepsilon_0$. For any multiindex $|\alpha| \le k$ and $0 < \varepsilon \le \varepsilon_0$,

$$\partial_x^\alpha u_\varepsilon = \partial_x^\alpha(\varphi_\varepsilon * u) = \varphi_\varepsilon * \partial^\alpha u \quad \text{on } \mathcal{V},$$

since $u \in C^k$ and $\varphi_\varepsilon \in C_c^\infty$ justify differentiation under the integral. Define, for $r \in [0, \varepsilon_0]$, the modulus of continuity

$$\omega_\alpha(r) := \sup_{\substack{x \in \mathcal{V} \\ |h| \le r}} \big|\partial^\alpha u(x-h) - \partial^\alpha u(x)\big|.$$

Because $\partial^\alpha u$ is uniformly continuous on $\{x \in \mathcal{U} : \operatorname{dist}(x, \mathcal{V}) \leq \varepsilon_0\}$, we have $\omega_\alpha(r) \to 0$ as $r \to 0$. Using $\operatorname{supp} \varphi_\varepsilon \subset B(0, \varepsilon)$ and $\int \varphi_\varepsilon = 1$,

$$\sup_{x \in \mathcal{V}} \left|\partial^\alpha u_\varepsilon(x) - \partial^\alpha u(x)\right| \leq \int_{B(0,\varepsilon)} \varphi_\varepsilon(y)\, \omega_\alpha(|y|)\, dy \leq \omega_\alpha(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus $\partial^\alpha u_\varepsilon \to \partial^\alpha u$ uniformly on $\mathcal{V}$, and since $\mathcal{V} \Subset \mathcal{U}$ was arbitrary, on every compact subset of $\mathcal{U}$.

4. Fix $\mathcal{V} \Subset \mathcal{U}$ and set $\varepsilon_0 := \frac{1}{2} \operatorname{dist}(\mathcal{V}, \partial\mathcal{U}) > 0$. It is enough to prove $\|\partial^\alpha u_\varepsilon - \partial^\alpha u\|_{L^p(\mathcal{V})} \to 0$. as $\varepsilon \to 0$, since summing over $|\alpha| \leq k$ yields $u_\varepsilon \to u$ in $W^{k,p}(\mathcal{V})$. For $0 < \varepsilon \leq \varepsilon_0$ and $|\alpha| \leq k$, using Fubini and the definition of weak derivatives, $\partial^\alpha u_\varepsilon = \partial^\alpha(\varphi_\varepsilon * u) = \varphi_\varepsilon * \partial^\alpha u$ on $\mathcal{V}$. For $f \in L^p_{\mathrm{loc}}(\mathcal{U})$, by Jensen's inequality (since $\varphi_\varepsilon \geq 0$ and $\int \varphi_\varepsilon = 1$),

$$\|\varphi_\varepsilon * f - f\|_{L^p(\mathcal{V})} = \left\| \int \varphi_\varepsilon(y)\big(\tau_{-y} f - f\big)\, dy \right\|_{L^p(\mathcal{V})} \leq \int \varphi_\varepsilon(y)\, \|\tau_{-y} f - f\|_{L^p(\mathcal{V})}\, dy.$$

Translations are continuous in L^p_{loc} (as for $p = 1$ above), hence the right-hand side $\to 0$ as $\varepsilon \to 0$ because $\operatorname{supp} \varphi_\varepsilon \subset B(0, \varepsilon)$. Apply this with $f = \partial^\alpha u$ for each $|\alpha| \leq k$ and use the identity above.

5. Fix $\delta > 0$. Decompose $\mathcal{U} = \cup_{\ell \geq 0} \mathcal{V}_\ell$ where we define

$$\mathcal{U}_\ell := \{x \in \mathcal{U} : \operatorname{dist}(x, \partial\mathcal{U}) > \ell^{-1}\}, \qquad \mathcal{V}_\ell := \mathcal{U}_{\ell+3} \setminus \overline{\mathcal{U}_{\ell+1}} \quad \text{for} \quad \ell \geq 1,$$

and choose $\mathcal{V}_0$ with $\mathcal{U} \setminus \bigcup_{\ell \geq 1} \mathcal{V}_\ell \subset \mathcal{V}_0 \Subset \mathcal{U}$. Let $(\xi_\ell)_{\ell \geq 0}$ be a smooth partition of unity subordinate to $(\mathcal{V}_\ell)$.[26] For each ℓ, pick $0 < \varepsilon_\ell < \frac{1}{2} \operatorname{dist}(\mathcal{V}_\ell, \partial\mathcal{U})$ such that, by (iv),

$$v_\ell := \varphi_{\varepsilon_\ell} * (\xi_\ell u) \in C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U}), \qquad \|v_\ell - \xi_\ell u\|_{W^{k,p}(\mathcal{U})} \leq \frac{\delta}{2^{\ell+1}}.$$

Here we are smoothing a localized version of u. Set $u^{(\delta)} := \sum_{\ell \geq 0} v_\ell$. The sum is locally finite, hence $u^{(\delta)} \in C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U})$, and

$$\|u - u^{(\delta)}\|_{W^{k,p}(\mathcal{U})} = \Big\| \sum_{\ell \geq 0} (\xi_\ell u - v_\ell) \Big\|_{W^{k,p}(\mathcal{U})} \leq \sum_{\ell \geq 0} \frac{\delta}{2^{\ell+1}} = \delta.$$

Choosing $\delta = 2^{-j}$ and $u_j := u^{(2^{-j})}$ yields $u_j \to u$ in $W^{k,p}(\mathcal{U})$.

[26] A smooth partition of unity subordinate to $(\mathcal{V}_\ell)_{\ell \geq 0}$ is a locally finite family $(\xi_\ell)_{\ell \geq 0} \subset C_c^\infty(\mathcal{U})$ with $0 \leq \xi_\ell \leq 1$, $\operatorname{supp} \xi_\ell \subset \mathcal{V}_\ell$ for each ℓ, and $\sum_{\ell \geq 0} \xi_\ell(x) \equiv 1$ for all $x \in \mathcal{U}$. Locally finite means every $x \in \mathcal{U}$ has a neighborhood meeting only finitely many $\operatorname{supp} \xi_\ell$, so the sum (and its derivatives) are well defined.

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6. *Step 1. Partition of unity.* Let $\mathcal{U} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Cover $\partial\mathcal{U}$ by finitely many boundary charts $\{B_\ell\}_{\ell=1}^N$ and one interior set $B_0 \Subset \mathcal{U}$, so that for $\ell \geq 1$ in local coordinates we have

$$\mathcal{U} \cap B_\ell = \{(x', t) \in B_\ell : \ t > \Gamma_\ell(x')\}, \qquad \Gamma_\ell \text{ Lipschitz with constant } \mathrm{Lip}(\Gamma_\ell) =: \mathfrak{L}_\ell.$$

Pick a smooth partition of unity $\{\psi_\ell\}_{\ell=0}^N$ with $\operatorname{supp}\psi_\ell \subset B_\ell$ and $\sum_{\ell=0}^N \psi_\ell \equiv 1$ on $\overline{\mathcal{U}}$.
Step 2. Mollification and translation on each piece. For the case $\ell = 0$, we just choose $\varepsilon > 0$ with $\varepsilon < \operatorname{dist}(\operatorname{supp}\psi_0, \partial\mathcal{U})$ and let ρ_ε be a standard mollifier. Then

$$\rho_\varepsilon * (\psi_0 u) \in C^\infty(\overline{\mathcal{U}}) \quad \text{and} \quad \rho_\varepsilon * (\psi_0 u) \to \psi_0 u \ \text{ in } W^{k,p}(\mathcal{U}) \text{ as } \varepsilon \to 0.$$

A plain convolution near $\partial\mathcal{U}$ would ask for points outside $\mathcal{U}$ to evaluate u, which is not allowed. Fix $\ell \in \{1, \dots, N\}$. Denote translations by $\tau_h u(x) := u(x+h)$. For $\varepsilon > 0$ small enough (so the balls of radius ε used below stay inside B_ℓ), define on $\mathcal{U} \cap B_\ell$

$$\big[\rho_\varepsilon * \tau_{\lambda_\ell \varepsilon e_n}(\psi_\ell u)\big](x) = \int_{\mathbb{R}^n} \rho_\varepsilon(y)\,(\psi_\ell u)\big(x - y + \lambda_\ell \varepsilon e_n\big)\,dy.$$

This is the usual convolution evaluated after the vertical translation $\tau_{\lambda_\ell \varepsilon e_n}$, i.e. a push into $\mathcal{U}$. The inward translation $\tau_{\lambda_\ell \varepsilon e_n}$ possibly avoids any exterior sampling, so we can smooth using only interior values. We must check that the ε–ball used by the standard mollifier stays inside $\mathcal{U}$ after this push.

Step 3. Checking that $\rho_\varepsilon * \tau_{\lambda_\ell \varepsilon e_n}(\psi_\ell u)$ requires evaluating u only on $\mathcal{U}$. Thanks to the Lipschitz boundary we have that there exists $\lambda_\ell \in \mathbb{R}$ so that

$$\forall\, x \in \partial\mathcal{U} \cap B(x_\ell, r_\ell),\ \forall\, \varepsilon > 0 \text{ small}, \qquad B\big(x + \lambda_\ell \varepsilon e_n,\ \varepsilon\big) \subset \mathcal{U}.$$

To see this, fix $(x', \Gamma_\ell(x')) \in \partial\mathcal{U} \cap B(x_\ell, r_\ell)$ and set $\lambda_\ell := \mathfrak{L}_\ell + 2$. For $\varepsilon > 0$ small, any $y = (y', y_n) \in B\big((x', \Gamma_\ell(x') + \lambda_\ell \varepsilon e_n), \varepsilon\big)$ satisfies

$$|y' - x'| < \varepsilon, \qquad y_n > \Gamma_\ell(x') + (\lambda_\ell - 1)\varepsilon.$$

Since Γ is Lipschitz we have

$$\Gamma_\ell(y') \leq \Gamma_\ell(x') + \mathfrak{L}_\ell |y' - x'| < \Gamma_\ell(x') + \mathfrak{L}_\ell \varepsilon,$$

hence

$$y_n - \Gamma_\ell(y') > (\lambda_\ell - 1 - \mathfrak{L}_\ell)\varepsilon = \varepsilon > 0.$$

Hence the whole ball $B\big((x', \Gamma_\ell(x') + \lambda_\ell \varepsilon e_n), \varepsilon\big)$ lies above the graph Γ_ℓ, this it is inside $\mathcal{U}$. Consequently, $\rho_\varepsilon * \tau_{\lambda_\ell \varepsilon e_n}(\psi_\ell u)$ only evaluates values of u inside $\mathcal{U}$ and

$$\rho_\varepsilon * \tau_{\lambda_\ell \varepsilon e_n}(\psi_\ell u) \in C^\infty(\overline{B_\ell \cap \mathcal{U}}).$$

Step 4. Construction of the global approximation. Set

$$u_\varepsilon \ := \ \rho_\varepsilon * (\psi_0 u) \ + \ \sum_{\ell=1}^N \rho_\varepsilon * \tau_{\lambda_\ell \varepsilon e_n}(\psi_\ell u) \ \in \ C^\infty(\overline{\mathcal{U}}).$$

Since $\sum_{\ell=0}^{N} \psi_\ell u = u$, it suffices to pass to the limit on each piece. The convergence in B_0 was noted above. Fix $\ell \geq 1$. Using mollification and the $W^{k,p}_{\text{loc}}$ continuity of translations (which follows from the L^p_{loc} case) we get

$$\|\rho_\varepsilon * \tau_{\lambda\varepsilon e_n}(\psi_\ell u) - \psi_\ell u\|_{W^{k,p}(\mathcal{U})} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.2}$$

Precisely, for any $|\alpha| \leq k$ write

$$D^\alpha\big(\rho_\varepsilon * \tau_{\lambda\varepsilon e_n}(\psi_\ell u) - \psi_\ell u\big) = A_{\varepsilon,\alpha} + B_{\varepsilon,\alpha},$$

where

$$A_{\varepsilon,\alpha} := \rho_\varepsilon * \tau_{\lambda\varepsilon e_n} D^\alpha(\psi_\ell u) - \tau_{\lambda\varepsilon e_n} D^\alpha(\psi_\ell u), \quad B_{\varepsilon,\alpha} := \tau_{\lambda\varepsilon e_n} D^\alpha(\psi_\ell u) - D^\alpha(\psi_\ell u).$$

Then $\|A_{\varepsilon,\alpha}\|_{L^p(\mathcal{U})} \to 0$ by mollification (part 4. of the proposition, with $k = 0$), and $\|B_{\varepsilon,\alpha}\|_{L^p(\mathcal{U})} \to 0$ by translation continuity (apply Leibniz rule, which follows from Exercise 2.5, to $D^\alpha(\psi_\ell u)$ and use that $D^\gamma\psi_\ell \in C_c^\infty$). Note that all these convergences are on $\operatorname{supp}\psi_\ell \Subset \mathcal{U}$. Summing over $|\alpha| \leq k$ gives (3.2).
Finally, summing over ℓ gives $\|u_\varepsilon - u\|_{W^{k,p}(\mathcal{U})} \to 0$; then choose $\varepsilon_j \to 0$ and set $u_j := u_{\varepsilon_j} \in C^\infty(\overline{\mathcal{U}})$.

□

3.5 Extensions and traces

In this section we extend Sobolev functions to larger domains with the Sobolev norm controlled by the original one. We also show that, unlike general L^p functions, Sobolev functions admit a well-defined *trace* (i.e. a restriction) on lower-dimensional, sets (of null Lebesgue measure).

Theorem 3.15 (Extension for $W^{1,p}$). *Let $\mathcal{U} \subset \mathbb{R}^n$ be a bounded C^1 domain and let $\mathcal{V} \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{U} \Subset \mathcal{V}$. For any $p \in [1,\infty)$ there exists a bounded linear* extension operator

$$E : W^{1,p}(\mathcal{U}) \to W^{1,p}(\mathbb{R}^n)$$

such that for every $u \in W^{1,p}(\mathcal{U})$,

$$E(u) = u \quad \textit{a.e. on } \mathcal{U}, \qquad \operatorname{supp} E(u) \subset \mathcal{V}.$$

Remark 3.16. Boundedness means there is $C = C(\mathcal{U}, \mathcal{V}, p) > 0$ with

$$\|E(u)\|_{W^{1,p}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{1,p}(\mathcal{U})} \quad \text{for all } u \in W^{1,p}(\mathcal{U}).$$

We call E the extension of u to $\mathbb{R}^n$. For (essential) support we mean, as for Lebesgue functions, $\operatorname{supp} E(u) := \mathbb{R}^n \setminus \bigcup\{\, U \subset \mathbb{R}^n \text{ open} : E(u) = 0 \text{ a.e. on } U\,\}$.

Proof. Step 1: Reflection Fix $x^0 \in \partial\mathcal{U}$ and assume $\partial\mathcal{U}$ is flat near x^0, i.e., there exists a ball B centered at x^0 with

$$\partial\mathcal{U} \cap B = \{x_n = 0\}, \qquad B^+ := B \cap \{x_n > 0\} \subset \mathcal{U}, \qquad B^- := B \cap \{x_n < 0\} \subset \mathbb{R}^n \setminus \mathcal{U}.$$

Assume $u \in C^1(\overline{B^+})$ and define the *higher order reflection* of $\tilde{u}$ from B^+ to B^- by

$$\tilde{u}(x', x_n) = \begin{cases} u(x', x_n), & x_n \geq 0, \\ -3\,u(x', -x_n) + 4\,u\Big(x', -\dfrac{x_n}{2}\Big), & x_n < 0, \end{cases} \tag{3.3}$$

where $x' = (x_1, \ldots, x_{n-1})$. Note that this extension is linear in u. Check from (3.3) that $\tilde{u}$ is continuous across $\{x_n = 0\}$. Then, we check that is C^1 across $\{x_n = 0\}$. Indeed, from the definition we see that the tangential derivatives are continuous, and for the normal derivative we get, for $x_n < 0$,

$$\partial_{x_n} \tilde{u}(x', x_n) = 3\, \partial_{x_n} u(x', -x_n) - 2\, \partial_{x_n} u\Big(x', -\frac{x_n}{2}\Big),$$

so, as $x_n \to 0^-$, it agrees with $\partial_{x_n} u(x', 0)$. Therefore $\tilde{u} \in C^1(B)$ and agrees with u on B^+. Using (3.3) and the chain rule one obtains

$$\|\tilde{u}\|_{W^{1,p}(B)} \le C\, \|u\|_{W^{1,p}(B^+)}, \tag{3.4}$$

for some $C = C(n,p)$ independent of u.

Step 2: Flattening the local C^1 boundary. Fix $x^0 \in \partial\mathcal{U}$. Since $\partial\mathcal{U}$ is C^1, there exist a ball B centered at x^0 and a C^1-diffeomorphism $\Phi : B \to \Phi(B)$ with C^1 inverse, such that, writing $y = \Phi(x)$ and $x = \Phi^{-1}(y)$,

$$\Phi(\partial\mathcal{U} \cap B) = \{y_n = 0\} \cap \Phi(B), \qquad \Phi(\mathcal{U} \cap B) = \{y_n > 0\} \cap \Phi(B).$$

Moreover, by shrinking B if necessary, we may assume

$$0 < c_0 \le |\det D\Phi(x)|,\ |\det D\Phi^{-1}(y)| \le C_0 \quad \text{and} \quad \|D\Phi\|_{L^\infty(B)} + \|D\Phi^{-1}\|_{L^\infty(\Phi(B))} \le M,$$

for some constants $c_0, C_0, M \ge 1$ depending only on the chart.

Let $u \in C^1(\overline{\mathcal{U} \cap B})$ and define

$$v(y) := u(\Phi^{-1}(y)) \quad \text{for } y \in \Phi(B) \cap \{y_n \ge 0\}.$$

Since the boundary is flat in y coordinates, apply Step 1 on $\Phi(B)$ to obtain an extension $\tilde{v} \in W^{1,p}(\Phi(B))$ with

$$\|\tilde{v}\|_{W^{1,p}(\Phi(B))} \ \le\ C_1\, \|v\|_{W^{1,p}\left(\Phi(B)\cap\{y_n>0\}\right)}, \tag{3.5}$$

where $C_1 = C_1(p,n)$. Pull back to x coordinates by setting

$$\tilde{u}(x) := \tilde{v}(\Phi(x)) \quad \text{for } x \in B.$$

From the chain rule we get the estimates

$$\|\tilde{u}\|^p_{L^p(B)} = \int_B |\tilde{v}(\Phi(x))|^p\, dx \le c_0^{-1} \int_{\Phi(B)} |\tilde{v}(y)|^p\, dy,$$
$$\|\nabla\tilde{u}\|^p_{L^p(B)} = \int_B \left|D\Phi(x)^\top\, \nabla\tilde{v}(\Phi(x))\right|^p dx \le c_0^{-1}\, \|D\Phi\|^p_{L^\infty(B)} \int_{\Phi(B)} |\nabla\tilde{v}(y)|^p\, dy,$$

hence

$$\|\tilde{u}\|_{W^{1,p}(B)} \ \le\ C_2\, \|\tilde{v}\|_{W^{1,p}(\Phi(B))}, \tag{3.6}$$

for some $C_2 = C_2(p, n, c_0, M)$. Similarly, for $v(y) = u(\Phi^{-1}(y))$ on $\Phi(B) \cap \{y_n > 0\}$,

$$\|v\|_{W^{1,p}\left(\Phi(B)\cap\{y_n>0\}\right)} \ \le\ C_3\, \|u\|_{W^{1,p}(\mathcal{U}\cap B)}, \tag{3.7}$$

for some $C_3 = C_3(p, n, C_0, M)$. Combining (3.5), (3.6), and (3.7) yields

$$\|\tilde{u}\|_{W^{1,p}(B)} \ \le\ \tilde{C}\, \|u\|_{W^{1,p}(\mathcal{U}\cap B)}, \qquad \tilde{C} := C_1\, C_2\, C_3, \tag{3.8}$$

where $\tilde{C} = \tilde{C}(p, n, c_0, C_0, M)$. Finally, note that $\tilde{u} = u$ on $\mathcal{U} \cap B$ by construction. This gives a local bounded linear extension on B.

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Step 3: Partition of unity Cover $\partial\mathcal{U}$ by finitely many balls $B_1, \ldots, B_N$. Also, choose $asetB_0 \Subset \mathcal{U}$ so that $\overline{\mathcal{U}} \subset B_0 \cup \bigcup_{i=1}^N B_i$. Take a partition of unity $\{\xi_i\}_{i=0}^N \subset C_c^\infty(B_i)$ subordinated to $\{B_i\}_{i=0}^n$ with $\sum \xi_i \equiv 1$ on $\overline{\mathcal{U}}$. For $u \in C^1(\overline{\mathcal{U}})$, extend each $\chi_i u$ for $i \geq 1$ to $E_i(\xi_i u)$ by Step 2 from $B_i \cap \mathcal{U}$ to B_i, and set

$$Eu := \sum_{i=0}^N E_i(\xi_i u),$$

where $E_0(\xi_0 u) = \xi u$ is just the identity from B_0 into itself. Choosing the cover inside $\mathcal{V}$ and shrinking supports if necessary gives $\operatorname{supp}(Eu) \subset \mathcal{V}$. Finite overlap and (3.8) give

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{1,p}(\mathcal{U})}.$$

Step 4: Density. Finally we remove the assumption $u \in C^1(\overline{\mathcal{U}})$ by a density argument. By the previous theorem (since $\partial\mathcal{U}$ being C^1 is in particular Lipschitz) there is $u_j \in C^\infty(\overline{\mathcal{U}})$ converging to u in $W^{1,p}(\mathcal{U})$. Since E is linear and bounded we deduce that Eu_j is Cauchy in $W^{1,p}(\mathcal{V})$ and therefore converges to some Eu with the expected bound. The limit does not depend on the approximation sequence since for two sequences we have

$$\left\|Eu_j^1 - Eu_j^2\right\|_{W^{1,p}(\mathcal{V})} = \left\|E\left(u_j^1 - u_j^2\right)\right\|_{W^{1,p}(\mathcal{V})} \leq C\left\|u_j^1 - u_j^2\right\|_{W^{1,p}(\mathcal{U})} \to 0 \quad \text{as} \quad j \to \infty$$

which concludes the proof. □

Remark 3.17. One can construct an extension operator in $W^{k,p}$ for $k \geq 2$. The argument is similar with an appropriate (and more complicated) reflection: in the flat case it has the form $\bar{v}(y) := \sum_{j=1}^k c_j v(\tilde{y}', -y_n'/j)$ on $B_- := B(y, r') \cap \{y_n' < 0\}$ with well-chosen coefficients c_j's.

Theorem 3.18 (Trace theorem). *Let $\mathcal{U} \subset \mathbb{R}^n$ be open, bounded, with C^1 boundary, and let $p \in [1, \infty)$. There exists a linear operator, the* trace operator*,*

$$T : W^{1,p}(\mathcal{U}) \longrightarrow L^p(\partial\mathcal{U})$$

that is bounded, i.e.

$$\|Tu\|_{L^p(\partial\mathcal{U})} \leq C\,\|u\|_{W^{1,p}(\mathcal{U})} \qquad \textit{for all } u \in W^{1,p}(\mathcal{U}),$$

where $C = C(n, p, \mathcal{U}) > 0$. Moreover, for every $u \in W^{1,p}(\mathcal{U}) \cap C^\infty(\overline{\mathcal{U}})$ one has $Tu = u|_{\partial\mathcal{U}}$.

Remark 3.19.

1. For $u \in C^\infty(\overline{\mathcal{U}})$, the trace coincides with the pointwise restriction $u|_{\partial\mathcal{U}}$. By density of $C^\infty(\overline{\mathcal{U}})$ in $W^{1,p}(\mathcal{U})$ (for $\partial\mathcal{U}$ being C^1), this identifies the trace uniquely.
2. If $u \in W^{k,p}(\mathcal{U})$, then traces exist for all derivatives up to order $k-1$: for each multiindex α with $|\alpha| \leq k-1$, one can define a bounded operator T_α so that $T_\alpha(D^\alpha u)$ is the trace of $D^\alpha u$ on $\partial\mathcal{U}$.
3. Zero trace characterisation: $u \in W_0^{1,p}(\mathcal{U})$ if and only if $Tu = 0$ in $L^p(\partial\mathcal{U})$. Indeed, if $u_j \in C_c^\infty(\mathcal{U})$ with $u_j \to u$ in $W^{1,p}(\mathcal{U})$, then $T(u_j) = 0$ and boundedness of T gives $Tu = 0$. The converse follows by localisation, flattening the boundary, and a partition of unity arguments (see Evans, Sec. 5.5, Thm 2).

4. Sharp regularity (for $1 < p < \infty$): if $s > 1/p$, there is, considering fractional derivatives (cf. Exercise 2.13) a bounded trace $T : W^{s,p}(\mathcal{U}) \longrightarrow W^{s-1/p,p}(\partial\mathcal{U})$, so the loss of differentiability is $1/p$. More generally, for traces onto a C^1 submanifold of codimension m, one requires $s > m/p$ and the loss is m/p.

Proof of Theorem 3.18. The proof follows the same structure as for the extension theorem, i.e. we construct the trace operator T by a covering argument, reducing locally to the flat case, using $u \in C^\infty(\overline{\mathcal{U}})$ and relaxing the latter by density. This reduces the proof to the flat whole space case (the partition of unity localises).

Let $B \subset \mathbb{R}^n$ be a ball centered on the hyperplane $\{x_n = 0\}$, and set $B^+ = B \cap \{x_n > 0\}$ and $\Gamma = \partial B^+ \cap \{x_n = 0\}$. Fix $\phi \in C_c^\infty(B)$ with $0 \le \phi \le 1$ and $\phi \equiv 1$ on a smaller ball $B' \Subset B$. For $u \in C^\infty(\overline{B^+})$ set $w := \phi u$ and $F := |w|^p e_n$. Since w vanishes near ∂B, using the divergence theorem and the Young's inequality we get

$$\begin{aligned}
\|u\|^p_{L^p(\Gamma\cap B')} &= \int_{\Gamma\cap B'} |u|^p \, d\sigma = \int_\Gamma |w|^p \, d\sigma = -\int_{\partial B^+} F\cdot\nu \, d\sigma = \int_{B^+} \nabla\cdot F \, dx = \int_{B^+} \partial_{x_n}(|w|^p) \, dx \\
&= \int_{B^+} p\,|w|^{p-1}\,\partial_{x_n} w \,\mathrm{sgn}(w) \, dx \;\le\; \int_{B^+} \Big((p-1)|w|^p + |\partial_{x_n} w|^p\Big) \, dx \\
&\le\; C\int_{B^+} \Big(|\phi u|^p + |(\partial_{x_n}\phi)u + \phi\,\partial_{x_n} u|^p\Big) \, dx \le\; C\int_{B^+} \Big(|u|^p + |\nabla u|^p\Big) \, dx \\
&\le C\,\|u\|^p_{W^{1,p}(B^+)} .
\end{aligned}$$

By density of $C^\infty(\overline{B^+})$ in $W^{1,p}(B^+)$, the estimate extends to all $u \in W^{1,p}(B^+)$. □

3.6 Sobolev inequalities

The Sobolev inequalities are a collection of inequalities that "trade" integrability of weak derivatives for classical differentiability. The basic result is the *Gagliardo-Nirenberg-Sobolev inequality (GNS)*. Before we state and prove it, we need the following (Loomis–Whitney) lemma.

Lemma 3.20. *Let $n \ge 2$ and $f_1, \dots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. For $x = (x_1, \dots, x_n)$ set $\tilde{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$. Then $f(x) := \prod_{i=1}^n f_i(\tilde{x}_i) \in L^1(\mathbb{R}^n)$ and*

$$\|f\|_{L^1(\mathbb{R}^n)} \le \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Proof. By replacing f_i with $|f_i|$, assume $f_i \ge 0$. For $n = 2$ we have $f(x_1, x_2) = f_1(x_2) f_2(x_1)$, hence $\|f\|_{L^1(\mathbb{R}^2)} = \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^1(\mathbb{R})}$. Assume the claim true in dimension $n \ge 2$ and consider $f_1, \dots, f_{n+1} \in L^n(\mathbb{R}^n)$. Fix $x_{n+1} \in \mathbb{R}$ and write $F(\cdot, x_{n+1}) = \prod_{i=1}^n f_i(\tilde{x}_i)$ and $f(\cdot, x_{n+1}) = f_{n+1}(\tilde{x}_{n+1})\, F(\cdot, x_{n+1})$. By Hölder on $\mathbb{R}^n$ with exponents n and $q = \frac{n}{n-1}$,

$$\int_{\mathbb{R}^n} |f(\cdot, x_{n+1})| \le \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \, \|F(\cdot, x_{n+1})\|_{L^q(\mathbb{R}^n)}.$$

Applying the inductive hypothesis in dimension n to $g_i := f_i(\cdot, x_{n+1})^q \in L^{n-1}(\mathbb{R}^{n-1})$ gives

$$\|F(\cdot, x_{n+1})\|_{L^q(\mathbb{R}^n)} \le \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}.$$

Therefore $\int_{\mathbb{R}^n} |f(\cdot, x_{n+1})| \le \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}$. Integrating in x_{n+1} and using Tonelli plus generalized Hölder (with n factors of exponent n), we obtain

$$\|f\|_{L^1(\mathbb{R}^{n+1})} \le \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^n \Big(\int_{\mathbb{R}} \|f_i(\cdot, x_{n+1})\|^n_{L^n(\mathbb{R}^{n-1})}\, dx_{n+1} \Big)^{1/n} = \prod_{i=1}^{n+1} \|f_i\|_{L^n(\mathbb{R}^n)}.$$

This is the desired estimate with n replaced by $n+1$, completing the induction. □

Theorem 3.21 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let $n \ge 2$ and $p \in [1, n)$. Write $p^* = \frac{np}{n-p}$ (equivalently, $1/p^* = 1/p - 1/n$). Then:*

- (Global, $\mathbb{R}^n$) *There exists $C = C(n, p) > 0$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\, \|Du\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } u \in W^{1,p}(\mathbb{R}^n).$$

- (Local away from the boundary) *If $\mathcal{U} \subset \mathbb{R}^n$ is open and bounded, there exists $C = C(\mathcal{U}, n, p) > 0$ such that*

$$\|u\|_{L^{p^*}(\mathcal{U})} \le C\, \|Du\|_{L^p(\mathcal{U})} \qquad \text{for all } u \in W^{1,p}_0(\mathcal{U}).$$

- (Local up to the boundary) *If $\mathcal{U} \subset \mathbb{R}^n$ is open, bounded with C^1 boundary, there exists $C = C(\mathcal{U}, n, p) > 0$ such that*

$$\|u\|_{L^{p^*}(\mathcal{U})} \le C\, \|u\|_{W^{1,p}(\mathcal{U})} \qquad \text{for all } u \in W^{1,p}(\mathcal{U}).$$

Remark 3.22.
1. Since $p^* = \frac{np}{n-p} > p$ for $p < n$, the embedding gives a genuine gain of integrability (evident in the local statements, where the L^p spaces are nested).
2. On $\mathbb{R}^n$, the estimate controls u only modulo additive constants via $\|Du\|_{L^p}$; the assumption $u \in W^{1,p}(\mathbb{R}^n)$ (in particular $u \in L^p$) rules out non decaying behaviours.
3. The $W^{1,p}_0(\mathcal{U})$ case, in particular gives $\|u\|_{L^p(\mathcal{U})} \le C\, \|Du\|_{L^p(\mathcal{U})}$, which is an instance of *Poincaré inequality.*

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Proof of Theorem 3.21. We first prove the global estimate on $\mathbb{R}^n$ assuming $u \in C_c^1(\mathbb{R}^n)$.
Step 1: the case $p = 1$. Fix $i \in \{1, \dots, n\}$. By the Fundamental Theorem of Calculus,

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \dots, y_i, \dots, x_n)\, dy_i,$$

so that $|u(x)| \le g_i(\tilde{x}_i)$ where

$$g_i(\tilde{x}_i) := \int_{-\infty}^{+\infty} \big|\partial_{x_i} u(x_1, \dots, y_i, \dots, x_n)\big|\, dy_i, \qquad \tilde{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}.$$

Let $f := |u|^{\frac{n}{n-1}}$ and $f_i := g_i^{\frac{1}{n-1}}$. Then $f(x) \le \prod_{i=1}^n f_i(\tilde{x}_i)$, and by Lemma 3.20,

$$\|f\|_{L^1(\mathbb{R}^n)} \le \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Therefore,

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}^{\frac{n}{n-1}} = \|f\|_{L^1(\mathbb{R}^n)} \le \prod_{i=1}^n \|g_i\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} = \prod_{i=1}^n \|\partial_{x_i} u\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n-1}} \le C_n \|\nabla u\|_{L^1(\mathbb{R}^n)}^{\frac{n}{n-1}}$$

for some C_n depending only on the dimension n. This gives

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_n \|\nabla u\|_{L^1(\mathbb{R}^n)} \qquad \forall\, u \in C_c^1(\mathbb{R}^n).$$

Step 2: the case $p \in (1, n)$. Let $\gamma := \dfrac{p(n-1)}{n-p} > 1$ and set $v := |u|^\gamma$. Since $u \in C_c^1(\mathbb{R}^n)$ and $\gamma > 1$, we have $v \in C_c^1(\mathbb{R}^n)$ with $\nabla v = \gamma |u|^{\gamma-1} \operatorname{sign}(u) \nabla u$. Applying the $p = 1$ estimate from Step 1 to v gives

$$\|u\|_{L^{\frac{pn}{n-p}}(\mathbb{R}^n)}^{\frac{p(n-1)}{n-p}} = \|v\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_n \|\nabla v\|_{L^1(\mathbb{R}^n)} = C_n \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |\nabla u|\, dx,$$

By Hölder with exponents $\left(\frac{p}{p-1}, p\right)$ and using $(\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p}$,

$$\int_{\mathbb{R}^n} |u|^{\gamma-1} |\nabla u| \le \big\| |u|^{\gamma-1} \big\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)} \|\nabla u\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^{\frac{pn}{n-p}}(\mathbb{R}^n)}^{\frac{n(p-1)}{n-p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

We obtain

$$\|u\|_{L^{\frac{pn}{n-p}}(\mathbb{R}^n)}^{\frac{p(n-1)}{n-p}} \le C_n \gamma \|u\|_{L^{\frac{pn}{n-p}}(\mathbb{R}^n)}^{\frac{n(p-1)}{n-p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

If $\|u\|_{L^{\frac{pn}{n-p}}} = 0$ there is nothing to prove; otherwise divide both sides by $\|u\|_{L^{\frac{pn}{n-p}}}^{\frac{n(p-1)}{n-p}}$ and get

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C_{n,p} \|\nabla u\|_{L^p(\mathbb{R}^n)}, \qquad p^* = \frac{np}{n-p}, \qquad \forall\, u \in C_c^1(\mathbb{R}^n).$$

Step 3: density and local versions. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$, the global inequality extends to all $u \in W^{1,p}(\mathbb{R}^n)$. Indeed, we can approximate by truncation and mollification: take

$u_j := (\eta_{R_j} u) * \rho_{\varepsilon_j}$ with cut-off $\eta_{R_j} \in C_c^\infty$, $\eta_{R_j} \equiv 1$ on $B(0, R_j)$, $R_j \to \infty$, and ρ_{ε_j} a mollifier with $\varepsilon_j \to 0$; then $u_j \to u$ in $W^{1,p}(\mathbb{R}^n)$.

Local away from the boundary. If $u \in C_c^\infty(\mathcal{U})$, let $\tilde{u}$ be its zero extension to $\mathbb{R}^n$. Then $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ and $\|\tilde{u}\|_{L^{p^*}(\mathbb{R}^n)} = \|u\|_{L^{p^*}(\mathcal{U})}$, $\|\nabla\tilde{u}\|_{L^p(\mathbb{R}^n)} = \|\nabla u\|_{L^p(\mathcal{U})}$. Applying the global inequality to $\tilde{u}$ yields $\|u\|_{L^{p^*}(\mathcal{U})} \le C_{n,p}\|\nabla u\|_{L^p(\mathcal{U})}$ for all $u \in C_c^\infty(\mathcal{U})$, and by density of $C_c^\infty(\mathcal{U})$ in $W_0^{1,p}(\mathcal{U})$ the estimate extends to every $u \in W_0^{1,p}(\mathcal{U})$.

Local up to the boundary. Thanks to the assumptions on $\mathcal{U}$, we can apply Theorem 3.15 giving the extension operator E. Applying the global inequality to Eu we obtain

$$\|u\|_{L^{p^*}(\mathcal{U})} \le \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \le C_{n,p}\,\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C_{n,p}\,C_{\text{ext}}\,\|u\|_{W^{1,p}(\mathcal{U})}.$$

□

Remark 3.23 (Critical case $p = n$). At $p = n$ (and $n \ge 2$) one does not get L^∞ control (as expected since $p^*(p) \to \infty$ as $p \to n$): e.g. $u(x) = \log\log\big(\frac{e}{|x|}\big)$ on $B(0, e^{-1}) \subset \mathbb{R}^n$ lies in $W^{1,n}B(0,e^{-1}) \setminus L^\infty B(0,e^{-1})$. It holds the endpoint embedding $W^{1,n}(\mathcal{U})$ into the space $\mathrm{BMO}(\mathcal{U})$ of functions of bounded mean oscillations, a space strictly between $\bigcap_{q<\infty} L^q$ and L^∞.

Remark 3.24 (1D case). The case $n = 1$ is excluded since we need $p \ge 1$ (to work with Banach spaces) but $p^* = \frac{np}{n-p}$ is finite only when $p < n$. In 1D, we get an L^∞ bound: for I finite interval and $u \in W_0^{1,p}(I)$ we have $\|u\|_{L^\infty(I)} \le C_{I,p}\|Du\|_{L^p(I)}$. On $\mathbb{R}$, if $u \in W^{1,1}(\mathbb{R})$ then $\|u\|_{L^\infty(\mathbb{R})} \le \|Du\|_{L^1(\mathbb{R})}$ (see the book by Brezis for more details on the 1D case).

With higher integrability of the function and its derivatives, specifically when $p > n$ we gain not only greater integrability and the expected boundedness but also pointwise regularity, as the following theorem shows.

Theorem 3.25 (Morrey inequality). *Let $p > n$ and $\gamma := 1 - \frac{n}{p}$.*

- (Global, $\mathbb{R}^n$) *There exists a constant $C = C(n,p) > 0$ such that for every $u \in C_c^\infty(\mathbb{R}^n)$,*

$$\|u\|_{C^{0,\gamma}(\overline{\mathbb{R}^n})} \le C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

  Thus, every $u \in W^{1,p}(\mathbb{R}^n)$ has a representative in $C^{0,\gamma}(\overline{\mathbb{R}^n})$ (i.e., there is a version of u, equal to u almost everywhere, that belongs to this Hölder space).
- (Local version up to the boundary) *If $\mathcal{U} \subset \mathbb{R}^n$ is bounded with C^1 boundary, then there exists a constant $C = C(\mathcal{U}, n, p) > 0$ such that for every $u \in C^\infty(\overline{\mathcal{U}})$,*

$$\|u\|_{C^{0,\gamma}(\overline{\mathcal{U}})} \le C\,\|u\|_{W^{1,p}(\mathcal{U})}.$$

  Thus, every $u \in W^{1,p}(\mathcal{U})$ has a representative in $C^{0,\gamma}(\overline{\mathcal{U}})$.

Proof. Global case. By density, prove it for $u \in C_c^\infty(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, $r > 0$, and any center m with $x \in B_r(m)$, set the average $\bar{u}_{m,r} := \frac{1}{|B_r|}\int_{B_r(m)} u$. By the Fundamental Theorem of Calculus,

$$u(x) - u(z) = \int_0^1 \nabla u\big(x + t(z-x)\big)\cdot(x-z)\,dt.$$

Averaging over $z \in B_r(m)$, using Fubini and $|z-x| \le 2r$

$$\begin{aligned}|u(x) - \bar{u}_{m,r}| &\le \frac{1}{|B_r|}\int_{B_r(m)}\int_0^1 \big|\nabla u\big(x+t(z-x)\big)\big|\,|z-x|\,dt\,dz \\ &\le \frac{2r}{|B_r|}\int_0^1\int_{B_r(m)} \big|\nabla u\big(x+t(z-x)\big)\big|\,dz\,dt.\end{aligned}$$

With the change of variables $y = x + t(z - x)$ we get

$$x + t\big(B_r(m) - x\big) = B_{tr}\big(x + t(m - x)\big) \subset B_r(m),$$

hence, using Hölder with $1 = \frac{1}{p} + \big(1 - \frac{1}{p}\big)$,

$$\begin{aligned}
|u(x) - \bar{u}_{m,r}| &\leq \frac{2r}{|B_r|} \int_0^1 t^{-n} \int_{B_{tr}(x+t(m-x))} |\nabla u(y)|\, dy\, dt \\
&\leq \frac{2r}{|B_r|} \int_0^1 t^{-n} \, |B_{tr}|^{1-\frac{1}{p}} \, \|\nabla u\|_{L^p(B_r(m))} \, dt \\
&= 2r \, |B_r|^{-\frac{1}{p}} \left(\int_0^1 t^{-\frac{n}{p}} \, dt \right) \|\nabla u\|_{L^p(B_r(m))} \\
&\leq C \, r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B_r(m))},
\end{aligned}$$

since $p > n$. Now take $m = \frac{x+y}{2}$ and $r = 2|x - y|$, so $x, y \in B_r(m)$:

$$|u(x) - u(y)| \leq |u(x) - \bar{u}_{m,r}| + |u(y) - \bar{u}_{m,r}| \leq C \, r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B_r(m))} \leq C \, |x - y|^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Also, we get the L^∞ bound. Set $m = x$ and $r = 1$, using the estimate above and Hölder

$$|u(x)| \leq |u(x) - \bar{u}_{x,r}| + |\bar{u}_{x,r}| \leq C \, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

This gives the claim for $C_c^\infty(\mathbb{R}^n)$, hence for $W^{1,p}(\mathbb{R}^n)$ by density.

Local case. From the assumptions on $\mathcal{U}$ Theorem 3.15 gives $E : W^{1,p}(\mathcal{U}) \to W^{1,p}(\mathbb{R}^n)$ with $Eu|_{\mathcal{U}} = u$ and $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C_{\text{ext}} \|u\|_{W^{1,p}(\mathcal{U})}$. Letting $u \in C^\infty(\overline{\mathcal{U}})$, we apply the global case to Eu:

$$\|u\|_{C^{0,\gamma}(\overline{\mathcal{U}})} \leq \|Eu\|_{C^{0,\gamma}(\overline{\mathbb{R}^n})} \leq C \, \|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \, C_{\text{ext}} \|u\|_{W^{1,p}(\mathcal{U})}.$$

By density of $C^\infty(\overline{\mathcal{U}})$ in $W^{1,p}(\mathcal{U})$ we get the local statement.

□

Remark 3.26. By applying the previous inequalities to u and its derivatives, one establishes higher order versions showing that functions in $W^{k,p}$ belongs to some L^q and/or $C^{\ell,\gamma}$ spaces. These are collectively the "Sobolev inequalities".

Example 3.27. If $u \in W^{2,2}(\mathcal{U})$ for $\mathcal{U} \subset \mathbb{R}^3$ we have $u, \nabla u \in W^{1,2}(\mathcal{U})$, thus Gagliardo-Nirenberg-Sobolev inequality gives $u, \nabla u \in L^6(\mathcal{U})$ and hence $u \in W^{1,6}(\mathcal{U})$. Applying Morrey inequality we get $u \in C^{0,1/2}(\overline{\mathcal{U}})$.

LECTURE 15

3.7 Compactness in Sobolev spaces

Theorem 3.28 (Rellich–Kondrachov compactness theorem). *Let $\mathcal{U} \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary and let $p \in [1, n)$. Set $p^* = \frac{np}{n-p}$. Then for every $q \in [1, p^*)$ the embedding $W^{1,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U})$ is compact; i.e., bounded sets in $W^{1,p}(\mathcal{U})$ are relatively compact in $L^q(\mathcal{U})$ (i.e. the closure is a compact subset of $L^q(\mathcal{U})$).*

In a metric space, relative compactness can be checked through sequences. Thus, rephrasing the statement: every bounded sequence in $W^{1,p}(\mathcal{U})$ has a subsequence that converges in $L^q(\mathcal{U})$.

Remark 3.29 (Compact vs. continuous embedding). Compactness implies continuity: if the inclusion $i : W^{1,p}(\mathcal{U}) \to L^q(\mathcal{U})$ is compact but not continuous, then i is unbounded, so there exist u_n with $\|u_n\|_{W^{1,p}} \leq 1$ and $\|i(u_n)\|_{L^q} \geq n$. However, compactness means i maps the unit ball into a relatively compact set, so $(i(u_n))$ has a convergent (hence bounded) subsequence, a contradiction.

Remark 3.30 (Rellich–Kondrachov in practice). On a bounded domain, uniform *a priori* bounds in a Sobolev space, for example $H^1(\mathcal{U})$, guarantee that from any approximating sequence we can extract a subsequence that converges in $L^2(\mathcal{U})$. This turns uniform estimates on approximations into actual limits, it reduces existence questions to proving such uniform bounds. Also, it will be crucial to prove the Fredholm alternative in Chapter 4.

We recall *Arzelà–Ascoli Theorem*: Let $K \subset \mathbb{R}^n$ be compact and $\mathcal{F} \subset C(K, \mathbb{R})$ be uniformly bounded and equicontinuous. Then $\mathcal{F}$ is relatively compact in $(C(K), \|\cdot\|_\infty)$; equivalently, every sequence in $\mathcal{F}$ admits a uniformly convergent subsequence on K.

Proof of Theorem 3.28. Step 1: Extension. Consider a bounded sequence (u_j) in $W^{1,p}(\mathcal{U})$. From the extension theorem there exists an extension $v_j := Eu_j$ bounded in $W^{1,p}(\mathbb{R}^n)$ with compact supports all included in a given bounded open set $\mathcal{V}$ with $\mathcal{U} \Subset \mathcal{V}$. Fix also an open set $\mathcal{W}$ with $\mathcal{V} \Subset \mathcal{W}$ and set $\varepsilon_0 := \operatorname{dist}(\overline{\mathcal{V}}, \partial\mathcal{W}) > 0$. Given (φ_ε) standard mollifiers, define $v_j^\varepsilon := \varphi_\varepsilon * v_j$. Then for every $0 < \varepsilon < \varepsilon_0$ one has $v_j^\varepsilon \in C_c^\infty(\mathcal{W})$ and $\operatorname{supp} v_j^\varepsilon \subset \mathcal{W}$ for all j. (Here and below, all norms are taken on the indicated set; since $\operatorname{supp} v_j \subset \mathcal{V}$, $L^p(\mathbb{R}^n)$ and $L^p(\mathcal{V})$-norms of v_j coincide.)

Step 2. Fix $1 \leq q < p^*$. We claim that

$$\sup_j \|v_j^\varepsilon - v_j\|_{L^q(\mathcal{V})} \xrightarrow[\varepsilon \to 0]{} 0.$$

By density, assume $v_j \in C_c^\infty(\mathcal{V})$; by Step 1, $\operatorname{supp} v_j \subset \mathcal{V}$ and $\sup_j \|\nabla v_j\|_{L^p(\mathcal{V})} < \infty$. With a standard mollifier (φ_ε),

$$|v_j^\varepsilon(x) - v_j(x)| = \Big|\int_{B(0,1)} \varphi(y)\big(v_j(x-\varepsilon y) - v_j(x)\big)\,dy\Big| \leq \varepsilon \int_{B(0,1)} \varphi(y) \int_0^1 |\nabla v_j(x - \varepsilon t y)|\,dt\,dy.$$

Integrating in x over $\mathcal{V}$ and using Fubini,

$$\|v_j^\varepsilon - v_j\|_{L^1(\mathcal{V})} \leq \varepsilon\,\|\nabla v_j\|_{L^1(\mathcal{V})} \leq C\,\varepsilon\,\|\nabla v_j\|_{L^p(\mathcal{V})},$$

where the last inequality uses the boundedness of $\mathcal{V}$ and Hölder. Interpolating between $L^1(\mathcal{V})$ and $L^{p^*}(\mathcal{V})$ with Hölder inequality, we have

$$\|f\|_{L^q(\mathcal{V})} \leq \|f\|_{L^1(\mathcal{V})}^\theta \|f\|_{L^{p^*}(\mathcal{V})}^{1-\theta}, \quad \text{with} \quad \theta = \frac{\frac{1}{q} - \frac{1}{p^*}}{1 - \frac{1}{p^*}} \in (0, 1).$$

Using the Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{R}^n$ (which applies to v_j^ε, v_j since they are compactly supported) and Young's inequality for convolution (because $\nabla v_j^\varepsilon = \varphi_\varepsilon * \nabla v_j$ and $\|\varphi_\varepsilon\|_{L^1} = 1$),

$$\begin{aligned}\|v_j^\varepsilon - v_j\|_{L^q(\mathcal{V})} &\le \|v_j^\varepsilon - v_j\|_{L^1(\mathcal{V})}^\theta \|v_j^\varepsilon - v_j\|_{L^{p^*}(\mathcal{V})}^{1-\theta} \\ &\le (C\varepsilon\|\nabla v_j\|_{L^p(\mathcal{V})})^\theta \left(\|v_j^\varepsilon\|_{L^{p^*}} + \|v_j\|_{L^{p^*}}\right)^{1-\theta} \\ &\le (C\varepsilon\|\nabla v_j\|_{L^p(\mathcal{V})})^\theta \left(C\|\nabla v_j^\varepsilon\|_{L^p} + C\|\nabla v_j\|_{L^p}\right)^{1-\theta} \\ &\le (C\varepsilon\|\nabla v_j\|_{L^p(\mathcal{V})})^\theta \left(C\|\nabla v_j\|_{L^p(\mathcal{V})}\right)^{1-\theta} = C\,\varepsilon^\theta\,\|\nabla v_j\|_{L^p(\mathcal{V})}.\end{aligned}$$

Since $\sup_j \|\nabla v_j\|_{L^p(\mathcal{V})} < \infty$, it follows that $v_j^\varepsilon \to v_j$ in $L^q(\mathcal{V})$ uniformly in j as $\varepsilon \to 0$.

Step 3. Compactness by Arzelà–Ascoli. Fix $\varepsilon \in (0, \varepsilon_0)$. For each such fixed $\varepsilon > 0$, (v_j^ε) is equibounded and equicontinuous on $\overline{\mathcal{W}}$:

$$\|v_j^\varepsilon\|_{L^\infty(\mathcal{W})} \le \|\varphi_\varepsilon\|_{L^\infty}\|v_j\|_{L^1(\mathcal{V})}, \qquad \|\nabla v_j^\varepsilon\|_{L^\infty(\mathcal{W})} \le \|\nabla\varphi_\varepsilon\|_{L^\infty}\|v_j\|_{L^1(\mathcal{V})},$$

and $\|v_j\|_{L^1(\mathcal{V})} \le C\,\|v_j\|_{L^p(\mathcal{V})}$. By Arzelà–Ascoli on the compact set $\overline{\mathcal{W}}$, for this fixed ε there is a subsequence $(v_{j_k}^\varepsilon)$ converging uniformly on $\overline{\mathcal{W}}$; hence, for any $\delta > 0$ and k, ℓ large enough,

$$\|v_{j_k}^\varepsilon - v_{j_\ell}^\varepsilon\|_{L^q(\mathcal{V})} < \delta/3.$$

Choose $\varepsilon \in (0, \varepsilon_0)$ so that, by Step 2,

$$\sup_j \|v_j^\varepsilon - v_j\|_{L^q(\mathcal{V})} < \delta/3.$$

Then for k, ℓ large,

$$\|v_{j_k} - v_{j_\ell}\|_{L^q(\mathcal{V})} \le \|v_{j_k} - v_{j_k}^\varepsilon\|_{L^q(\mathcal{V})} + \|v_{j_k}^\varepsilon - v_{j_\ell}^\varepsilon\|_{L^q(\mathcal{V})} + \|v_{j_\ell}^\varepsilon - v_{j_\ell}\|_{L^q(\mathcal{V})} < \delta,$$

so (v_{j_k}) is Cauchy in $L^q(\mathcal{V})$, hence convergent in $L^q(\mathcal{V})$ to some $v \in L^q(\mathcal{V})$. Finally, since $v_j = Eu_j$ and $v_j = u_j$ a.e. on $\mathcal{U}$, by restriction we obtain

$$u_{j_k} = v_{j_k}\big|_{\mathcal{U}} \longrightarrow v\big|_{\mathcal{U}} \quad \text{in } L^q(\mathcal{U}).$$

□

Remark 3.31 (Failure at the critical exponent). The embedding is not compact at the critical exponent $p^* = \frac{np}{n-p}$ for $1 \le p < n$. To see it, take any nonzero $u \in C_c^\infty(B(0,r))$ with $B(0,r) \subset \mathcal{U}$ and define $v_\varepsilon(x) := \varepsilon^{-n/p^*} u(x/\varepsilon)$ for $0 < \varepsilon \le 1$. Then $\operatorname{supp} v_\varepsilon \subset B(0, \varepsilon r) \subset \mathcal{U}$ and $\|\nabla v_\varepsilon\|_{L^p} = \|\nabla u\|_{L^p}$, $\|v_\varepsilon\|_{L^{p^*}} = \|u\|_{L^{p^*}} > 0$. Hence $\{v_\varepsilon\}$ is bounded in $W^{1,p}(\mathcal{U})$ and has a constant $\|v_\varepsilon\|_{L^{p^*}} > 0$, while $v_\varepsilon(x) \to 0$ for every fixed $x \ne 0$, so $v_\varepsilon \to 0$ almost everywhere. If the embedding $W^{1,p}(\mathcal{U}) \to L^{p^*}(\mathcal{U})$ were compact, some subsequence would converge in L^{p^*}. Passing, if necessary, to a subsequence of that subsequence, we may assume $\|v_{\varepsilon_k}\|_{L^{p^*}} \to 0$, contradicting $\|v_{\varepsilon_k}\|_{L^{p^*}} = \|u\|_{L^{p^*}} > 0$.

4 Ellipticity

We always consider $\mathcal{U} \subset \mathbb{R}^n$ open bounded and with C^1 boundary in this chapter.

4.1 The notion of ellipticity

At the most general level, we have seen in Chapter 2 that ellipticity corresponds for a linear operator to the absence of characteristic surfaces, i.e. $\sigma_d(x,\xi) \neq 0$ for all $x \in \mathcal{U}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. We now write it concretely for second-order linear operators. Such operators L can be given in two forms, given $a_{ij}, b_i, \tilde{b}_i : \mathcal{U} \to \mathbb{R}$,

$$Lu = \underbrace{-\sum_{i=1}^n \partial_{x_i}\left(\sum_{j=1}^n a_{ij}\partial_{x_j}u\right)}_{-\nabla\cdot(ADu)} + \sum_{i=1}^n b_i \partial_{x_i}u + cu \qquad \text{(divergence form)}$$

$$Lu = -\sum_{i,j=1}^n a_{ij}\,\partial^2_{x_i x_j}u + \sum_{i=1}^n \tilde{b}_i \partial_{x_i}u + cu \qquad \text{(non-divergence form)}$$

One can navigate between divergence-form and non-divergence form whenever a_{ij} is differentiable by $\tilde{b}_i = b_i - \sum_{j=1}^n \partial_{x_j} a_{ji}$. Note also, supposing $\partial_{x_i}\partial_{x_j}u = \partial_{x_j}\partial_{x_i}u$, we can assume that the matrix A with entries (a_{ij}) is symmetric by modifying if necessary the first order term: if we decompose $a_{ij} = a^s_{ij} + a^{as}_{ij}$ in symmetric and anti-symmetric part, where $a^s_{ij} := \frac{a_{ij}+a_{ji}}{2}$ and $a^{as}_{ij} := \frac{a_{ij}-a_{ji}}{2}$, we obtain

$$\sum_{i=1}^n \partial_{x_i}\left(\sum_{j=1}^n a_{ij}\partial_{x_j}u\right) = \sum_{i=1}^n \partial_{x_i}\left(\sum_{j=1}^n a^s_{ij}\partial_{x_j}u\right),$$

$$\sum_{i,j=1}^n a_{ij}\,\partial^2_{x_i x_j}u = \sum_{i,j=1}^n a^s_{ij}\,\partial^2_{x_i x_j}u.$$

Definition 4.1. We say that L is **elliptic** if $\sum_{i,j=1}^n a^{ij}(x)\,\xi_i\xi_j > 0$ for all $x \in \mathcal{U}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, and we say that L is **uniformly elliptic** (also called **strictly elliptic**) if for some $\theta > 0$ we have $\sum_{i,j=1}^n a^{ij}(x)\,\xi_i\xi_j > \theta|\xi|^2$ for all $x \in \mathcal{U}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

The goal of the chapter is to study the equation $Lu = f$ for some source term f, in bounded domains $\mathcal{U}$, and for some uniformly elliptic second-order linear operator L. As we have previously noted with Hadamard's example, the Cauchy Problem for elliptic operators, even if we can apply the Cauchy-Kovalevskaya theorem to solve it locally and uniquely in the analytic class, is ill-posed in the C^k class (and also, with the same mechanism, in Sobolev spaces). Thus, we relax the data constraints and only impose a boundary condition on u, namely $u|_{\partial\mathcal{U}}$ will be given. We will define a setting where we can construct a unique solution depending continuously on the data, and then we will consider the regularity theory of such solutions.

LECTURE 16

4.2 Solving the Dirichlet problem (coercive case)

Consider the boundary value problem (the **Dirichlet problem**)

$$\begin{cases} Lu = f & \text{in } \mathcal{U}, \\ u = 0 & \text{on } \partial\mathcal{U}, \end{cases} \tag{4.1}$$

where L is the divergence form operator

$$Lu = -\sum_{i,j=1}^{n} \partial_{x_i}\big(a_{ij}\,\partial_{x_j} u\big) + \sum_{i=1}^{n} b_i\,\partial_{x_i} u + c\,u,$$

with bounded measurable coefficients. If $u \in C^2(\overline{\mathcal{U}})$ solves (4.1), then for any $v \in C^1(\overline{\mathcal{U}})$ with $v|_{\partial\mathcal{U}} = 0$, integrating by parts (the boundary terms vanish) gives

$$\langle f, v\rangle_{L^2(\mathcal{U})} = \int_{\mathcal{U}} f\,v = \int_{\mathcal{U}} \left[\sum_{i,j=1}^{n} a_{ij}\,(\partial_{x_j} u)(\partial_{x_i} v) + \sum_{i=1}^{n} b_i\,(\partial_{x_i} u)\,v + c\,u\,v \right] =: B[u,v]. \tag{4.2}$$

Conversely, if $u \in C^2(\overline{\mathcal{U}})$ with $u|_{\partial\mathcal{U}} = 0$ satisfies (4.2), then for every $v \in C_c^\infty(\mathcal{U})$ another integration by parts yields $\langle Lu - f, v\rangle_{L^2(\mathcal{U})} = 0$, hence, by the fundamental lemma of calculus of variations, $Lu = f$ in $\mathcal{U}$. Note that (4.2) makes sense for $u, v \in H_0^1(\mathcal{U})$; the boundary condition is encoded by the trace.

Definition 4.2. Let $f \in L^2(\mathcal{U})$. We say that $u \in H_0^1(\mathcal{U})$ is a **weak solution** to (4.1) if

$$\forall v \in H_0^1(\mathcal{U}), \qquad B[u,v] = \langle f, v\rangle, \tag{4.3}$$

where B is defined by (4.2).

Proposition 4.3. *Assume $f \in L^2(\mathcal{U})$ and $a_{ij}, b_i, c \in L^\infty(\mathcal{U})$. Let $u \in C^2(\overline{\mathcal{U}}) \cap H_0^1(\mathcal{U})$. Then u is a weak solution to (4.1) if and only if*

$$Lu = f \quad \text{a.e. in } \mathcal{U}, \qquad u = 0 \quad \text{on } \partial\mathcal{U}.$$

If, in addition, $a_{ij} \in C^1(\overline{\mathcal{U}})$ and $b_i, c, f \in C(\overline{\mathcal{U}})$, then the identity $Lu = f$ holds pointwise in $\mathcal{U}$, so u is a classical solution.

Remark 4.4. This expresses the minimal requirement for a generalization of the notion of solution: it agrees with the classical one whenever sufficient regularity is available.

Proof of Proposition 4.3. ($\Leftarrow$) Assume $Lu = f$ a.e. in $\mathcal{U}$ and $u = 0$ on $\partial\mathcal{U}$. For $v \in C_c^\infty(\mathcal{U})$, multiplying the PDE by v and integrating by parts (no boundary term) gives

$$\int_{\mathcal{U}} f\,v = \int_{\mathcal{U}} \left[\sum_{i,j=1}^{n} a_{ij}\,(\partial_{x_j} u)(\partial_{x_i} v) + \sum_{i=1}^{n} b_i\,(\partial_{x_i} u)\,v + c\,u\,v \right] = B[u,v].$$

By density of $C_c^\infty(\mathcal{U})$ in $H_0^1(\mathcal{U})$ this extends to all $v \in H_0^1(\mathcal{U})$, hence u is a weak solution.

($\Rightarrow$) Assume u is a weak solution: for all $v \in H_0^1(\mathcal{U})$, $\int_{\mathcal{U}} f\,v = B[u,v]$. For $v \in C_c^\infty(\mathcal{U})$, by integration by parts we get $\int_{\mathcal{U}} (Lu - f)\,v = 0$ for all $v \in C_c^\infty(\mathcal{U})$, thus by the fundamental lemma of calculus of variation we get $Lu = f$ in $\mathcal{U}$ almost everywhere. Finally, $u \in H_0^1(\mathcal{U})$ has trace 0 in $L^2(\partial\mathcal{U})$; since $u \in C(\overline{\mathcal{U}})$, $u \equiv 0$ on $\partial\mathcal{U}$ pointwise. ☐

The main tool for existence and uniqueness is the following generalization of Riesz representation theorem.

Theorem 4.5 (Lax–Milgram)**.** *Let H be a real Hilbert space with inner product $(\cdot,\cdot)$, and $B : H \times H \to \mathbb{R}$ a bilinear form that is bounded and coercive:*

$$|B[u,v]| \le \alpha\, \|u\|_H \|v\|_H, \qquad B[u,u] \ge \beta\, \|u\|_H^2 \quad \text{for some } \alpha, \beta > 0.$$

Then for every $F \in H^*$ there exists a unique $u \in H$ such that $B[u,\cdot] = F(\cdot)$, and

$$\|u\|_H \le \beta^{-1}\, \|F\|_{H^*}, \quad \text{where} \quad \|F\|_{H^*} := \sup_{v \in H \setminus \{0\}} \frac{|F(v)|}{\|v\|_H}$$

We first apply the theorem to the Dirichlet problem, then prove it.

Remark 4.6 (An equivalent norm on H_0^1). We have proved as special case of Gagliardo-Nirenberg-Sobolev inequality that

$$\|u\|_{L^2(\mathcal{U})} \le C_P\, \|\nabla u\|_{L^2(\mathcal{U})} \qquad \forall\, u \in H_0^1(\mathcal{U}).$$

In particular, if we set

$$\|u\|_{H_0^1(\mathcal{U})} := \|\nabla u\|_{L^2(\mathcal{U})},$$

then $\|\cdot\|_{H^1(\mathcal{U})}$ and $\|\cdot\|_{H_0^1(\mathcal{U})}$ are equivalent norms on $H_0^1(\mathcal{U})$.

Corollary 4.7. *Assume L is in divergence form, uniform elliptic, with bounded measurable coefficients a_{ij}, with $b_i = 0$, $c \ge 0$. Then Lax–Milgram applies with $H = H_0^1(\mathcal{U})$, B as in* (4.2)*, and $F(v) = \int_{\mathcal{U}} f\, v$ for $f \in L^2(\mathcal{U})$. Consequently, there exists a unique $u \in H_0^1(\mathcal{U})$ solving* (4.3)*, and for some $C > 0$*

$$\|u\|_{H_0^1(\mathcal{U})} \le C\, \|f\|_{L^2(\mathcal{U})}.$$

Proof of Corollary 4.7. (1) *Boundedness of B.* By the boundedness of the coefficients and Cauchy–Schwarz, (here the controlling constant C can vary)

$$|B[u,v]| \le C\|\nabla u\|_{L^2}\, \|\nabla v\|_{L^2} + c\|u\|_{L^2}\, \|v\|_{L^2} \le C\|\nabla u\|_{L^2}\, \|\nabla v\|_{L^2} = C\|u\|_{H_0^1(\mathcal{U})}\, \|v\|_{H_0^1(\mathcal{U})}.$$

(2) *Coercivity.* With $b_i = 0$ and $c \ge 0$,

$$B[u,u] = \int_{\mathcal{U}} \sum_{i,j=1}^n a_{ij}\, (\partial_{x_j} u)(\partial_{x_i} u) + \int_{\mathcal{U}} c\, u^2 \;\ge\; \theta \int_{\mathcal{U}} |\nabla u|^2 = \theta\, \|u\|_{H_0^1(\mathcal{U})}^2.$$

(3) *Boundedness of F.* For $v \in H_0^1(\mathcal{U})$,

$$|F(v)| = \Big| \int_{\mathcal{U}} f\, v \Big| \le \|f\|_{L^2(\mathcal{U})}\, \|v\|_{L^2(\mathcal{U})} \lesssim \|f\|_{L^2(\mathcal{U})}\, \|\nabla v\|_{L^2(\mathcal{U})} = \|f\|_{L^2(\mathcal{U})}\, \|v\|_{H_0^1(\mathcal{U})}.$$

The claim follows from Theorem 4.5, with $\|u\|_{H_0^1} \le \theta^{-1}\, \|F\|_{(H_0^1)^*} \le \theta^{-1}\, \|f\|_{L^2(\mathcal{U})}$. □

We recall the following basic theorem in the theory of Hilbert spaces.

Theorem 4.8 (Riesz representation theorem)**.** *Let H be a Hilbert space with inner product $(\cdot,\cdot)$. For every continuous linear functional $F \in H^*$ there exists a unique $w \in H$ such that*

$$F(v) = (w, v) \qquad \forall\, v \in H,$$

and $\|F\|_{H^*} = \|w\|_H$.

Proof of Theorem 4.5. Fix $u \in H$, then functional $v \mapsto B[u,v]$ is linear and bounded, hence by Riesz there is a unique $Au \in H$ with

$$B[u,v] = (Au, v) \qquad \forall\, v \in H.$$

This defines a bounded linear operator $A : H \to H$ since, letting $\|\cdot\| = \|\cdot\|_H$ for simplicity,

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\| \;\Rightarrow\; \|Au\| \leq \alpha \|u\|.$$

Coercivity gives

$$\beta \|u\|^2 \leq B[u,u] = (Au, u) \leq \|Au\| \|u\| \;\Rightarrow\; \|Au\| \geq \beta \|u\|,$$

so A is injective and has closed range, indeed if (Au_n) is Cauchy, then (u_n) is Cauchy and converges to some u, hence by continuity of A we have $Au_n \to Au$. Moreover, if $v \perp \mathrm{Ran}(A)$ then $(Au, v) = 0$ for all u, so in particular $B[v,v] = (Av, v) = 0$, and coercivity forces $v = 0$. Recall that for any subspace $M \subset H$ we have the decomposition $H = \overline{M} \oplus (\overline{M})^\perp$, where $S^\perp$ is the orthogonal subspace of S. Thus $\mathrm{Ran}(A)$ is closed and has trivial orthogonal, hence $\mathrm{Ran}(A) = H$ and A is bijective. From $\|Au\| \geq \beta \|u\|$ we get $\|A^{-1}\|_{H^*} \leq \beta^{-1}$. Given $F \in H^*$, let $w \in H$ be its Riesz representative, $F(v) = (w, v)$ for all $v \in H$. The unique solution to $B[u,v] = F(v)$ is $u = A^{-1}w$, indeed $F(v) = (w,v) = (Au, v) = B[u,v]$, and $\|u\| \leq \|A^{-1}\| \|w\| = \beta^{-1} \|F\|_{H^*}$, using Riesz again. □

Remark 4.9. When B is symmetric, the weak formulation (4.3) is the Euler–Lagrange equation of the strictly convex functional $J(u) := \frac{1}{2}B[u,u] - F(u)$ on $H^1_0(\mathcal{U})$. Also, in this case the proof of Lax-Milgram is much easier: $\langle \cdot, \cdot \rangle_B := B[\cdot, \cdot]$ is an equivalent inner product for H, thus we can apply Riesz to $(H, \langle \cdot, \cdot \rangle_B)$ to immediately get Lax-Milgram for symmetric.

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4.3 Solving the Dirichlet problem (degenerate coercive case)

We consider L in divergence form, uniformly elliptic and with $a_{ij}, b_i, c \in L^\infty(\mathcal{U})$. The associated bilinear form

$$B[u,v] = \int_{\mathcal{U}} \Big(\sum_{i,j=1}^n a_{ij}\, \partial_{x_i} u\, \partial_{x_j} v + \sum_{i=1}^n b_i\, \partial_{x_i} u\, v + c\, u\, v \Big)\, dx$$

satisfies the **energy estimate**[27]

$$\forall u \in H_0^1(\mathcal{U}), \qquad B[u,u] \;\geq\; \beta \|u\|^2_{H_0^1(\mathcal{U})} - \gamma \|u\|^2_{L^2(\mathcal{U})} \tag{4.4}$$

for some $\beta > 0$ and $\gamma \geq 0$ (the case of Corollary 4.7 corresponds to $\gamma = 0$). Indeed, uniform ellipticity (of the symmetric part of $A = (a_{ij})$) gives (using Cauchy-Schwarz and Young's inequality)

$$\begin{aligned} B[u,u] &= \int_{\mathcal{U}} \sum_{i,j=1}^n a_{ij}\, \partial_{x_i} u\, \partial_{x_j} u + \sum_{i=1}^n \int_{\mathcal{U}} b_i\, \partial_{x_i} u\, u + \int_{\mathcal{U}} c\, u^2 \\ &\geq \theta \|\nabla u\|^2_{L^2(\mathcal{U})} - \bar{b}\sqrt{n}\, \|\nabla u\|_{L^2(\mathcal{U})} \|u\|_{L^2(\mathcal{U})} - \bar{c}\|u\|^2_{L^2(\mathcal{U})} \\ &\geq \tfrac{\theta}{2} \|\nabla u\|^2_{L^2(\mathcal{U})} - \Big(\tfrac{n\bar{b}^2}{2\theta} + \bar{c} \Big) \|u\|^2_{L^2(\mathcal{U})}, \end{aligned} \tag{4.5}$$

with $\bar{b} := \max_i \|b_i\|_{L^\infty}$ and $\bar{c} := \|c\|_{L^\infty}$. Since implies (4.4) with $\beta = \frac{\theta}{2}$ and $\gamma = \frac{n\bar{b}^2}{2\theta} + \bar{c}$.

By (4.4), $Lu + \mu u = f$ has a unique solution $u \in H_0^1(\mathcal{U})$ for every $\mu \geq \gamma$, since the weak form

$$B_\mu[u,v] := B[u,v] + \mu(u,v)_{L^2(\mathcal{U})}$$

satisfies Lax–Milgram.

To solve the problem in the degenerate coercive case, that is when $B[u,v]$ is not coercive but, as seen above, we still have (4.5), we will suitably reformulate it in terms of an abstract problem of the form $(\mathrm{Id} - K)u = h$ in the Hilbert space L^2, with $K : H \to H$ compact, that means the following.

Definition 4.10. Let H be a Hilbert space. A bounded operator $K : H \to H$ is **compact** if every bounded sequence (u_m) has a subsequence (Ku_{m_k}) that converges in H.

Remark 4.11. If $\mathcal{U}$ is bounded with C^1 boundary and $T : L^2(\mathcal{U}) \to H_0^1(\mathcal{U})$ is bounded, then the composition $K := \iota \circ T : L^2(\mathcal{U}) \to L^2(\mathcal{U})$ is compact, where $\iota : H_0^1(\mathcal{U}) \hookrightarrow L^2(\mathcal{U})$ is the (Rellich–Kondrachov) compact embedding.

Definition 4.12 (Adjoint operator)**.** Let H be a Hilbert space and $K : H \to H$ a bounded linear operator. The *adjoint* of K is the unique bounded linear operator $K^* : H \to H$ such that

$$\langle Kx, y \rangle = \langle x, K^* y \rangle \quad \text{for all } x, y \in H.$$

[27] For elliptic operators, the quadratic form $u \mapsto B[u,u]$ plays the role of an energy (compare $B[u,u] = \int_{\mathcal{U}} |\nabla u|^2$ for $L = -\Delta$). The inequality (4.4) (in literature also called Gårding inequality) shows that this energy controls the H_0^1–norm (up to lower-order L^2 terms), hence the name "energy estimate".

Remark 4.13. If K is compact than K^* is compact as well (check it).

The main property that we will use is that operators of the form $Id - K$, with K compact, are "almost invertible": their kernel and cokernel $H \setminus \operatorname{Im}(Id - K)$ are finite dimensional. More precisely, we have the following theorem.

Theorem 4.14 (Fredholm alternative for compact operators). *Let H be a real Hilbert space and $K : H \to H$ a compact operator. Then*

(i) $\operatorname{Ker}(Id - K)$ *is finite dimensional;*

(ii) $\operatorname{Im}(Id - K)$ *is closed;*

(iii) $\operatorname{Im}(Id - K) = \operatorname{Ker}(Id - K^*)^\perp$*;*

(iv) $\operatorname{Ker}(Id - K) = \{0\}$ *if and only if* $\operatorname{Im}(Id - K) = H$*;*

(v) $\dim \operatorname{Ker}(Id - K) = \dim \operatorname{Ker}(Id - K^*)$.

Remark 4.15. This motivates the notion of a Fredholm operator: a bounded linear map $T\colon X \to Y$ is *Fredholm* if $\operatorname{Ker} T$ and $Y/\operatorname{Im} T$ are finite dimensional and $\operatorname{Im} T$ is closed; its "Fredholm index" is defined as $\operatorname{ind} T := \dim \operatorname{Ker} T - \operatorname{codim} \operatorname{Im} T$. For $T = \mathrm{Id} - K$ with K compact one always has $\operatorname{ind} T = 0$.

Definition 4.16 (Adjoint of L). On $L^2(\mathcal{U})$ we define the (formal) adjoint L^* of L by

$$(Lu, v)_{L^2(\mathcal{U})} \;=\; B[u, v] \;=\; (u, L^*v)_{L^2(\mathcal{U})} \quad \text{for all } u, v \in C_c^\infty(\mathcal{U}).$$

For

$$Lu = -\sum_{i,j=1}^{n} \partial_i\big(a_{ij}(x)\,\partial_j u\big) + \sum_{i=1}^{n} b_i(x)\,\partial_i u + c(x)\,u,$$

corresponds to

$$L^*v = -\sum_{i,j=1}^{n} \partial_j\big(a_{ij}(x)\,\partial_i v\big) - \sum_{i=1}^{n} \partial_i\big(b_i(x)\,v\big) + c(x)\,v.$$

We now reformulate the Dirichlet problem in terms of $(\mathrm{Id} - K)u = h$ and apply the previous theorem.

Corollary 4.17 (Fredholm alternative, divergence form). *Let $\mathcal{U} \subset \mathbb{R}^n$ be bounded with C^1 boundary. Assume L is a divergence-form, uniformly elliptic operator with bounded coefficients, and let $f \in L^2(\mathcal{U})$. Consider*

$$\begin{cases} Lu = f & \text{in } \mathcal{U}, \\ u = 0 & \text{on } \partial\mathcal{U}. \end{cases}$$

Let L^* be the formal adjoint of L (as in the definition above). Exactly one of the following holds:

1. *There is a unique weak solution $u \in H_0^1(\mathcal{U})$.*
2. *The homogeneous problem has a nontrivial solution. Writing*

$$\mathsf{N} := \{u \in H_0^1(\mathcal{U}) : Lu = 0\}, \qquad \mathsf{N}^* := \{v \in H_0^1(\mathcal{U}) : L^*v = 0\},$$

we have $\dim \mathsf{N} = \dim \mathsf{N}^* < \infty$, *and $Lu = f$ is solvable if and only if*

$$\int_{\mathcal{U}} f\,\phi\,dx = 0 \quad \textit{for all } \phi \in \mathsf{N}^* \ (\textit{i.e. } f \perp \mathsf{N}^* \textit{ in } L^2).$$

When solvable, the solution set is the affine space $u + \mathsf{N}$ for any particular solution u.

Remark 4.18. The statement resembles the resolution of the matrix equation $Ax = b$.

Proof of Corollary 4.17. Let $B[\,\cdot\,,\,\cdot\,]$ be the bilinear form of L on $H_0^1(\mathcal{U})$ and let $(\,\cdot\,,\,\cdot\,)$ denote the $L^2(\mathcal{U})$-inner product. By the energy estimate

$$B[u,u] + \mu\,\|u\|_{L^2}^2 \;\geq\; \beta\|u\|_{H_0^1}^2 \qquad (u \in H_0^1(\mathcal{U}))$$

for some fixed $\mu > 0$ (referring to (4.5) take $\mu \geq \gamma$), Lax–Milgram gives, for every $g \in L^2(\mathcal{U})$, a unique $w \in H_0^1(\mathcal{U})$ solving

$$B[w,\phi] + \mu\,(w,\phi) = (g,\phi) \qquad \forall\,\phi \in H_0^1(\mathcal{U}),$$

with $\|w\|_{H_0^1} \leq \beta^{-1}\|g\|_{L^2}$. Define the bounded "resolvent"

$$R_\mu : L^2(\mathcal{U}) \to H_0^1(\mathcal{U}), \qquad R_\mu g = w,$$

and let $i : H_0^1(\mathcal{U}) \hookrightarrow L^2(\mathcal{U})$ be the compact Sobolev embedding (Rellich–Kondrachov). Set

$$K := \mu\, i \circ R_\mu \;:\; L^2(\mathcal{U}) \to L^2(\mathcal{U}).$$

Then, by Remark 4.11, K is compact on $H := L^2(\mathcal{U})$.
Step 1: Reformulation The Dirichlet problem $Lu = f$ in $H_0^1(\mathcal{U})$ rewrites as

$$B[u,\phi] = (f,\phi) \quad \forall\,\phi \in H_0^1(\mathcal{U}).$$

We claim that for $u \in L^2(\mathcal{U})$

$$(\mathrm{Id} - K)u = \mu^{-1}Kf \quad \text{if and only if} \quad u \in H_0^1(\mathcal{U}) \text{ and } B[u,\cdot] = (f,\cdot)\;,$$

namely u is a weak solution of $Lu = f$.
Indeed, if $(\mathrm{Id} - K)u = \mu^{-1}Kf$, then $u = K\big(u + \mu^{-1}f\big) = \mu\, i\, R_\mu\big(u + \mu^{-1}f\big)$, so $u \in i(H_0^1)$ and hence $u \in H_0^1(\mathcal{U})$. Put $z := R_\mu\big(u + \mu^{-1}f\big) \in H_0^1$. By definition of R_μ,

$$B[z,\phi] + \mu\,(z,\phi) = (u + \mu^{-1}f, \phi) \qquad \forall\,\phi \in H_0^1(\mathcal{U}).$$

Since $u = \mu\, i z$, we have $(u,\phi) = \mu\,(z,\phi)$, hence $B[z,\phi] = \mu^{-1}(f,\phi)$ and therefore $B[u,\phi] = (f,\phi)$ for all ϕ.
Conversely, if $u \in H_0^1$ and $B[u,\phi] = (f,\phi)$, then $B[u,\phi] + \mu(u,\phi) = (f + \mu u, \phi)$, so by the definition of R_μ, $u = R_\mu(f + \mu u)$. Applying i and multiplying by μ, $\mu\, i(R_\mu(f + \mu u)) = \mu\, i(u)$, i.e. $K(f + \mu u) = \mu\, u$, which is equivalent to $(\mathrm{Id} - K)u = \mu^{-1}Kf$.

Step 3: Apply Theorem 4.14. If $u \in \mathrm{Ker}\,(\mathrm{Id} - K)$, then by Step 1 with $f = 0$, $u \in H_0^1(\mathcal{U})$ and $B[u,\cdot] = 0$, i.e. u is a homogeneous weak solution: $\mathrm{Ker}\,(\mathrm{Id} - K) = \mathsf{N}$. Likewise, $\mathrm{Ker}\,(\mathrm{Id} - K^*) = \mathsf{N}^*$, the space of homogeneous weak solutions of the adjoint problem.
By Theorem 4.14 applied to K on $H = L^2(\mathcal{U})$:

- If $\mathsf{N} = \mathrm{Ker}\,(\mathrm{Id} - K) = \{0\}$, then (by (iv)) $\mathrm{Im}\,(\mathrm{Id} - K) = H$, so $(\mathrm{Id} - K)u = \mu^{-1}Kf$ has a unique solution $u \in L^2$. By Step 1 this is the unique weak solution $u \in H_0^1(\mathcal{U})$ of $Lu = f$.
- Otherwise, $\mathsf{N} = \mathrm{Ker}\,(\mathrm{Id} - K) \neq \{0\}$ and by (iii)

$$\mathrm{Im}\,(\mathrm{Id} - K) = \mathrm{Ker}\,(\mathrm{Id} - K^*)^\perp = (\mathsf{N}^*)^\perp.$$

Hence $(\mathrm{Id} - K)u = \mu^{-1}Kf$ is solvable if and only if $Kf \perp \mathsf{N}^*$ in $L^2(\mathcal{U})$. But for every $v \in \mathsf{N}^* = \mathrm{Ker}\,(\mathrm{Id} - K^*)$ we have $K^*v = v$, so

$$(Kf, v) = (f, K^*v) = (f, v),$$

and therefore $Kf \perp \mathsf{N}^*$ if and only if $f \perp \mathsf{N}^*$. Thus $Lu = f$ is solvable if and only if $\int_{\mathcal{U}} f\,\phi\,dx = 0$ for all $\phi \in \mathsf{N}^*$. Finally, by (v), $\dim \mathsf{N} = \dim \mathsf{N}^* < \infty$, and the solution set is the affine space $u + \mathsf{N}$.

□

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Remark 4.19. The Fredholm alternative implies in particular that for the Dirichlet problem above the following two statements are equivalent:

(a) for every $f \in L^2(\mathcal{U})$ there exists at least one weak solution $u \in H_0^1(\mathcal{U})$;
(b) for every $f \in L^2(\mathcal{U})$ there exists at most one weak solution $u \in H_0^1(\mathcal{U})$.

Indeed, (b) is equivalent to $\mathsf{N} = \mathrm{Ker}\,(\mathrm{Id} - K) = \{0\}$. By Theorem 4.14 this implies $\mathsf{N}^* = \mathrm{Ker}\,(\mathrm{Id} - K^*) = \{0\}$ and hence $\mathrm{Im}\,(\mathrm{Id} - K) = \mathrm{Ker}\,(\mathrm{Id} - K^*)^\perp = H = L^2(\mathcal{U})$, which is (a). Conversely, (a) says $\mathrm{Im}\,(\mathrm{Id} - K) = H$, so $\mathrm{Ker}\,(\mathrm{Id} - K^*) = \{0\}$ and then $\mathsf{N} = \mathrm{Ker}\,(\mathrm{Id} - K) = \{0\}$ by $\dim \mathsf{N} = \dim \mathsf{N}^*$, giving (b).
This is the infinite dimensional analogue of the fact that a linear map between finite-dimensional spaces of the same dimension is injective if and only if it is surjective.

As for the matrix equation $Ax = b$, it is crucial to understand the directions that A only stretches (eigenvectors) and the corresponding factors (eigenvalues) λ which may be complex. By analogy, for our elliptic operator L we are led to complexify (to not missing eigenvalues, using that $\mathbb{C}$ is an algebraically closed field) and look for $\lambda \in \mathbb{C}$ such that the operator $L - \lambda I$ is not invertible.

Remark 4.20 (Complexification). We complexify and look at the Dirichlet problem for $u, f : \mathcal{U} \to \mathbb{C}$. The L^2 and H_0^1 complex Hilbert space are defined with the corresponding inner products

$$(u, v)_{L^2} = \int_{\mathcal{U}} \overline{u}\, v\, dx, \qquad (u, v)_{H_0^1} = \int_{\mathcal{U}} \overline{\nabla u} \cdot \nabla v\, dx.$$

All previous results and proofs carry over assuming complex uniform ellipticity

$$\Re\Big(\sum_{i,j=1}^{n} a_{ij}(x)\, \overline{\xi_i}\, \xi_j \Big) \geq \theta |\xi|^2.$$

and $\Re(B[\cdot,\cdot]) \geq \beta \|u\|_{L^2}^2$ replacing coercivity.

We can then shift by $\lambda \in \mathbb{C}$ and ask for which λ the problem $(L - \lambda)u = f$ is uniquely solvable for all $f \in L^2(\mathcal{U})$. The set of such λ is the resolvent set, whose complement is the spectrum Σ of L.

Theorem 4.21 (Spectrum of L). *Let L be a divergence-form, uniformly elliptic operator on $\mathcal{U}$ with bounded coefficients $a_{ij}, b_i, c \in L^\infty(\mathcal{U})$ and consider the Dirichlet condition with $u|_{\partial\mathcal{U}} = 0$. Consider the (weak formulation) of the eigenvalue problem*

$$(L - \lambda)u = f,$$

for $f \in L^2(\mathcal{U})$, $u \in H_0^1(\mathcal{U})$ and $\lambda \in \mathbb{C}$. Then:

1. *There exists an at most countable set $\Sigma \subset \mathbb{C}$ such that for every $\lambda \notin \Sigma$ and every $f \in L^2(\mathcal{U})$ there is a unique weak solution $u \in H_0^1(\mathcal{U})$.*
2. *If Σ is infinite, writing $\Sigma = \{\lambda_k\}_{k \geq 1}$, we have $|\lambda_k| \to \infty$.*
3. *For each $\lambda \in \Sigma$, λ is an eigenvalue with finite dimensional eigenspace*

$$\mathcal{E}(\lambda) := \big\{ u \in H_0^1(\mathcal{U}) :\ B[u, \phi] = \lambda\, (u, \phi)_{L^2} \ \ \forall\, \phi \in H_0^1(\mathcal{U}) \big\} \neq \{0\}.$$

4. *If $a_{ij} = \overline{a_{ji}}$, $b_i = 0$, and c is real-valued, then L is self-adjoint with compact resolvent; in particular*

$$\Sigma \subset (\, \operatorname*{ess\,inf}_{\mathcal{U}} c,\ +\infty) \subset \mathbb{R}.$$

Proof of Theorem 4.21. Step 1: Reformulation. Fix $\mu \geq \gamma$ (where γ is given by the energy estimate (4.5)) and define $L_\mu := L + \mu I$. The shifted form $B_\mu[u,\varphi] := B[u,\varphi] + \mu(u,\varphi)_{L^2}$ is coercive on $H_0^1(\mathcal{U})$, so by Lax–Milgram, for each $f \in L^2(\mathcal{U})$ there exists a unique $u \in H_0^1(\mathcal{U})$ such that

$$B_\mu[u,\varphi] = (f,\varphi)_{L^2} \quad \forall \varphi \in H_0^1(\mathcal{U}).$$

This defines a bounded inverse $L_\mu^{-1} : L^2(\mathcal{U}) \to H_0^1(\mathcal{U})$. Let $i : H_0^1(\mathcal{U}) \hookrightarrow L^2(\mathcal{U})$ be the compact embedding. Then

$$T := i \circ L_\mu^{-1} : L^2(\mathcal{U}) \to L^2(\mathcal{U})$$

is a compact operator. If $u \in H_0^1(\mathcal{U})$ and $f \in L^2(\mathcal{U})$, the equation $(L-\lambda)u = f$ is equivalent to $L_\mu u = (\mu+\lambda)u + f$. Applying L_μ^{-1} and then i gives

$$(I - (\mu+\lambda)T)u = Tf. \tag{4.6}$$

In particular, any weak solution of $(L-\lambda)u = f$ lies in $H_0^1(\mathcal{U})$.
Step 2: Eigenvalues and eigenspaces (part (iii)). Let Σ be the set of $\lambda \in \mathbb{C}$ for which $(L-\lambda)u = 0$ admits a nontrivial weak solution $u \in H_0^1(\mathcal{U})$. If $\lambda \in \Sigma$, there exists $u \neq 0$ with $(L-\lambda)u = 0$. Plugging $f = 0$ into (4.6) we get

$$(I - (\mu+\lambda)T)u = 0.$$

Thus u is an eigenvector of T with nonzero eigenvalue $\nu = \frac{1}{\mu+\lambda}$. Conversely, if $Tu = \nu u$ with $\nu \neq 0$, then inserting in (4.6) with $f = 0$ gives $(I - (\mu+\lambda)T)u = 0$ which is equivalent to $(L-\lambda)u = 0$, where $\lambda = \nu^{-1} - \mu$. Hence $\lambda \in \Sigma$ if and only if $\nu = 1/(\mu+\lambda)$ is a nonzero eigenvalue of T, and the corresponding eigenspace

$$\mathcal{E}(\lambda) := \left\{ u \in H_0^1(\mathcal{U}) : B[u,\varphi] = \lambda(u,\varphi)_{L^2} \ \forall \varphi \in H_0^1(\mathcal{U}) \right\}$$

is exactly the eigenspace of T for ν. Since T is compact on the Hilbert space $L^2(\mathcal{U})$, every nonzero eigenspace of T is finite-dimensional. Thus each $\lambda \in \Sigma$ has a finite-dimensional eigenspace $\mathcal{E}(\lambda)$, which proves part (iii).

Assume, for some $M > 0$, that $\Sigma \cap B(0,M)$ is infinite, and let $(\lambda_k)_{k\geq 1} \subset \Sigma \cap B(0,M)$ be pairwise distinct. For each k choose $u_k \in H_0^1(\mathcal{U}) \setminus \{0\}$ such that $(L-\lambda_k)u_k = 0$ and normalize so that $\|u_k\|_{L^2} = 1$. As in Step 2, u_k is an eigenvector of the compact operator $T = i \circ L_\mu^{-1}$ with eigenvalue

$$\nu_k := \frac{1}{\mu+\lambda_k}.$$

The eigenvalues ν_k are pairwise distinct and satisfy

$$|\nu_k| = \frac{1}{|\mu+\lambda_k|} \geq \frac{1}{\mu+|\lambda_k|} \geq \frac{1}{\mu+M} =: \delta > 0.$$

Let

$$X := \overline{\operatorname{span}}\{u_k : k \geq 1\} \subset L^2(\mathcal{U}).$$

Since the u_k are linearly independent, X is infinite-dimensional. Moreover $T(X) \subset X$, so $S := T|_X : X \to X$ is a compact operator (on the Hilbert space X).

We claim that S is invertible and that its inverse is bounded with $\|S^{-1}\| \leq 1/\delta$. Indeed, every $x \in X$ can be written (in the sense of finite sums, and then by density) as $x = \sum_{k=1}^{N} c_k u_k$ we we get

$$Sx = Tx = \sum_{k=1}^{N} c_k \nu_k u_k.$$

Thus S acts diagonally in the basis $\{u_k\}$, with $Su_k = \nu_k u_k$. Since each $\nu_k \neq 0$, S is injective and its range contains each u_k (because $u_k = \nu_k^{-1} S u_k$), hence $\operatorname{Ran} S$ contains $\operatorname{span}\{u_k\}$. $\operatorname{Ran} S$ is also closed (as $\|Sx\| \geq \delta \|x\|$), we have $\operatorname{Ran} S = X$, so S is bijective. On the finite linear span of $\{u_k\}$ we have

$$\left\|S^{-1}\Big(\sum_{k=1}^{N} c_k \nu_k u_k\Big)\right\| = \left\|\sum_{k=1}^{N} c_k u_k\right\| \leq \frac{1}{\delta}\left\|\sum_{k=1}^{N} c_k \nu_k u_k\right\|,$$

since $|\nu_k| \geq \delta$. Hence $\|S^{-1}\| \leq 1/\delta$ on this dense subspace, and by continuity $S^{-1} : X \to X$ is bounded with $\|S^{-1}\| \leq 1/\delta$. Therefore the identity on X factorizes as $I_X = S^{-1} \circ S$, a composition of a bounded operator S^{-1} with a compact operator S. Thus I_X is compact on X. But X is infinite-dimensional, and the identity operator cannot be compact: take an orthonormal basis of X as sequence, to see that, even if it is bounded, it does not possess convergent subsequence. This contradiction shows that $\Sigma \cap B(0, M)$ must be finite.
Part (iv) Assume now that $a_{ij} = \overline{a_{ji}}$, $b_i = 0$, and c is real-valued. Then we actually have $B[u, u] = \overline{B[u, u]}$, thus $B[u, u]$ is real. Then from $B[u, u] = \lambda \|u\|^2$ we get that λ is also real. and from uniform ellipticity we get

$$\lambda = \frac{B[u, u]}{\|u\|^2_{L^2(\mathcal{U})}} \geq \theta \frac{\|\nabla u\|^2}{\|u\|^2} + \operatorname*{ess\,inf}_{\mathcal{U}} c$$

that, with the Poincaré inequality $\|u\|_{L^2} \leq C_P \|\nabla u\|_{L^2}$ for some $C_P > 0$, proves in particular $\Sigma \subset (\operatorname{ess\,inf}_{\mathcal{U}} c, \infty)$. □

Remark 4.22.

1. From the proof we also obtain the *resolvent estimate*

   $$\|L_\mu^{-1}\|_{L^2(\mathcal{U}) \to L^2(\mathcal{U})} \leq C(1 + \mu)^{-1}$$

   for $\mu \geq \gamma + 1$ and some $C = C(\gamma)$. Indeed, if $u = L_\mu^{-1} f$ then testing the weak formulation with u and using the energy estimate gives

   $$\theta \|\nabla u\|^2_{L^2} - \gamma \|u\|^2_{L^2} \leq \Re B[u, u] + \mu \|u\|^2_{L^2} = \Re(f, u)_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2}.$$

   Hence $(\mu - \gamma)\|u\|^2_{L^2} \leq \|f\|_{L^2}\|u\|_{L^2}$ and $\|L_\mu^{-1}\|_{L^2 \to L^2} \leq (\mu - \gamma)^{-1}$. For $\mu \geq \gamma + 1$, $(\mu - \gamma)^{-1} \leq (\gamma + 2)(1 + \mu)^{-1}$, which yields the stated estimate.
   When L is self-adjoint with spectrum $\Sigma \subset \mathbb{R}$, one can get a sharper bound $\|(L - \lambda)^{-1}\|_{L^2 \to L^2} = d(\lambda, \Sigma)^{-1}$ for every $\lambda \notin \Sigma$ (see Reed & Simon book Methods of Modern Mathematical Physics, Vol I) (*not examinable*).
2. In the model case $L = -\Delta$ then $\Sigma = \{\lambda_k\} \subset \mathbb{R}_+$ and these are the **harmonic frequencies** of the domain $\mathcal{U}$, at which a $\mathcal{U}$-shaped drum vibrates.
3. An interesting question, raised by Schuster, Bers and made famous in the 1966 article "*Can One Hear the Shape of a Drum?*" by Mark Kac, is whether the sequence of eigenvalues uniquely determines the shape of $\mathcal{U}$. In dimension 2, it does for convex analytic domains (Zelditch 2000), but not for some concave polygons (Gordon, Webb and Volpert 1992).

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4.4 Regularity theory of the Dirichlet problem

The goal of this section is to improve the regularity of the solution we have constructed in $H_0^1(\mathcal{U})$ so that it satisfies the strong (not just weak) formulation. We also obtain C^∞–regularity under appropriate assumptions on the coefficients of L and on $\mathcal{U}$. The core elliptic estimate, in the simplest case $-\Delta u = f$ with $u \in C_c^\infty(\mathcal{U})$, is

$$\begin{aligned}\int_{\mathcal{U}} f^2 = \int_{\mathcal{U}} (\Delta u)^2 = \int_{\mathcal{U}} \Big(\sum_{i=1}^n D_{ii}u\Big)^2 = \int_{\mathcal{U}} \sum_{i,j=1}^n D_{ii}u\, D_{jj}u \\ = \int_{\mathcal{U}} \sum_{i,j=1}^n D_{ij}u\, D_{ij}u = \int_{\mathcal{U}} \left|D^2 u\right|^2,\end{aligned} \tag{4.7}$$

where the last equality follows from two integrations by parts and the fact that u has compact support in $\mathcal{U}$. Thus the L^2–norm of the full Hessian is controlled by (indeed equal to) the L^2–norm of its trace, and $\|u\|_{H^2}$ is controlled by the datum $\|f\|_{L^2}$.

Theorem 4.23 (Interior elliptic regularity). *Let L be uniformly elliptic in divergence form, with bounded coefficients and $\mathcal{U}$ bounded. Let $k \geq 2$, $a_{ij}, b_i, c \in C^{k-1}(\mathcal{U})$, and $f \in H^{k-2}_{\text{loc}}(\mathcal{U})$. If $u \in H^1(\mathcal{U})$ satisfies the weak formulation $B[u,\cdot] = (f,\cdot)$, then $u \in H^k_{\text{loc}}(\mathcal{U})$. More precisely, for any $\mathcal{V} \subset\subset \tilde{\mathcal{W}} \subset\subset \mathcal{U}$ there exists $C > 0$ (depending on $k, \mathcal{V}, \tilde{\mathcal{W}}, \mathcal{U}, a, b, c, n$) such that*

$$\|u\|_{H^k(\mathcal{V})} \leq C\Big(\|f\|_{H^{k-2}(\tilde{\mathcal{W}})} + \|u\|_{L^2(\tilde{\mathcal{W}})}\Big).$$

Remark 4.24. This shows that the solution is strong: $u \in H^2(\mathcal{V})$ implies $Lu = f$ holds a.e. in $\mathcal{V}$ by testing against $v \in C_c^\infty(\mathcal{V})$. By Sobolev embedding (Theorem 3.21), if $m > n/2$ then $H^{m+2}(\mathcal{V}) \subset C^2(\mathcal{V})$. Hence, if $a_{ij}, b_i, c \in C^{m+1}(\mathcal{U})$ and $f \in H^m(\mathcal{U})$ with such m, then $u \in C^2_{\text{loc}}(\mathcal{U})$ and $Lu = f$ holds in the classical sense on $\mathcal{U}$. If the coefficients and f are smooth, then u is smooth (locally).

To get the result we would like to test the weak formulation with $D_{\ell\ell}u$ (similarly to what we have done in (4.7)), but we do not have second derivatives for the moment. To overcome this difficulty we work with a discretized derivative, namely the **difference quotients** and study their properties in connection with the weak derivative.

Proof of Theorem 4.23. It suffices to prove the case $k = 2$, the higher-order case following by induction. Indeed, assume the estimate is known for some integer $m \geq 2$. Suppose $a_{ij}, b_i, c \in C^m(\mathcal{U})$, $f \in H^{m-1}_{\text{loc}}(\mathcal{U})$, and u is a weak solution. By the induction hypothesis with $k = m$ we obtain $u \in H^m_{\text{loc}}(\mathcal{U})$. For each ℓ the weak derivative $\tilde{u} = D_\ell u$ satisfies

$$B[\tilde{u}, \varphi] = (\tilde{f}, \varphi) \quad \forall\, \varphi \in C_c^\infty(\mathcal{U}), \qquad \tilde{f} := D_\ell f - D_j\big[(\partial_\ell a_{ij}) D_i u\big] - (\partial_\ell b_i) D_i u - (\partial_\ell c)\, u,$$

which is obtained by testing the weak formulation for u with $v = -D_\ell \varphi$ and integrating by parts. Since $a_{ij}, b_i, c \in C^m$ and $u \in H^m_{\text{loc}}$, it follows that $\tilde{f} \in H^{m-2}_{\text{loc}}(\mathcal{U})$, so applying the $k = m$ estimate to $\tilde{u}$ yields $\tilde{u} \in H^m_{\text{loc}}(\mathcal{U})$ and hence $u \in H^{m+1}_{\text{loc}}(\mathcal{U})$ with the stated bound. Thus it remains to prove the case $k = 2$.

For $i \in \{1, \dots, n\}$ and $|h| < d(\mathcal{V}, \partial\mathcal{U})$ define the difference quotient

$$\Delta_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}, \qquad x \in \mathcal{V} \subset\subset \mathcal{U}.$$

For $u \in H^1(\mathcal{U})$ we have

$$D(\Delta_i^h u) = \Delta_i^h (Du).$$

Moreover, for $\mathcal{V} \Subset \tilde{\mathcal{W}} \Subset \mathcal{U}$ there exists $C > 0$, independent of h, such that

$$\|\Delta_i^h u\|_{L^2(\mathcal{V})} \leq C \|D_i u\|_{L^2(\tilde{\mathcal{W}})} \quad \text{for all } |h| \text{ small,} \tag{4.8}$$

and conversely if we have the uniform bound $\|\Delta_i^h u\|_{L^2(\mathcal{V})} \leq C$ as $h \to 0$ then $D_i u \in L^2(\mathcal{V})$ and $\|D_i u\|_{L^2(\mathcal{V})} \leq C$ (with the same constant) holds. Fix also $\mathcal{W}$ such that $\mathcal{V} \Subset \mathcal{W} \Subset \tilde{\mathcal{W}} \Subset \mathcal{U}$ and let $\psi \in C_c^\infty(\mathcal{W})$ satisfy $\psi \equiv 1$ on $\mathcal{V}$. For each $\ell \in \{1, \dots, n\}$ test the weak formulation with[28]

$$v := -\Delta_\ell^{-h}\big(\psi^2 \, \Delta_\ell^h u\big) \in H_0^1(\tilde{\mathcal{W}}), \qquad |h| < d(\mathcal{W}, \partial\tilde{\mathcal{W}}),$$

so that

$$I_a + I_b + I_c = I_f,$$

where[29]

$$I_a := \int_{\mathcal{U}} a_{ij} D_i u \, D_j \Delta_\ell^{-h}(\psi^2 \Delta_\ell^h u), \quad I_b := \int_{\mathcal{U}} b_i D_i u \, \Delta_\ell^{-h}(\psi^2 \Delta_\ell^h u),$$
$$I_c := \int_{\mathcal{U}} c \, u \, \Delta_\ell^{-h}(\psi^2 \Delta_\ell^h u), \quad I_f := \int_{\mathcal{U}} f \, \Delta_\ell^{-h}(\psi^2 \Delta_\ell^h u).$$

Estimate for I_b. Using the discrete integration-by-parts identities

$$\int_{\tilde{\mathcal{W}}} w \, \Delta_\ell^{-h} \zeta = -\int_{\tilde{\mathcal{W}}} \Delta_\ell^h w \, \zeta,$$

valid for w, ζ supported in $\mathcal{W} \subset\subset \tilde{\mathcal{W}}$ and $|h| < d(\mathcal{W}, \partial\tilde{\mathcal{W}})$, we get

$$\begin{aligned}
|I_b| &= \left| \int_{\tilde{\mathcal{W}}} b_i D_i u \, \Delta_\ell^{-h}(\psi^2 \Delta_\ell^h u) \right| = \left| -\int_{\tilde{\mathcal{W}}} \psi^2 \, \Delta_\ell^h (b_i D_i u) \, \Delta_\ell^h u \right| \\
&= \left| -\int_{\tilde{\mathcal{W}}} \psi^2 \big[(\tau_{he_\ell} b_i) \, \Delta_\ell^h D_i u + (\Delta_\ell^h b_i) \, D_i u \big] \Delta_\ell^h u \right| \\
&\leq C \int_{\tilde{\mathcal{W}}} \psi^2 \big(|\Delta_\ell^h Du| + |Du| \big) |\Delta_\ell^h u| \\
&\leq C \Big(\int_{\tilde{\mathcal{W}}} \psi^2 |\Delta_\ell^h Du|^2 \Big)^{1/2} \|\Delta_\ell^h u\|_{L^2(\tilde{\mathcal{W}})} + C \|Du\|_{L^2(\tilde{\mathcal{W}})} \|\Delta_\ell^h u\|_{L^2(\tilde{\mathcal{W}})} \\
&\leq C \Big(\|u\|_{H^1(\tilde{\mathcal{W}})}^2 + \|u\|_{H^1(\tilde{\mathcal{W}})} \Big(\int_{\tilde{\mathcal{W}}} \psi^2 |D \Delta_\ell^h u|^2 \Big)^{1/2} \Big) \\
&\leq \varepsilon \int_{\tilde{\mathcal{W}}} \psi^2 |D \Delta_\ell^h u|^2 + C_\varepsilon \|u\|_{H^1(\tilde{\mathcal{W}})}^2,
\end{aligned}$$

[28] Observe that this choice is dictated by the fact that for $u \in C^2$ we have $\Delta_\ell^{-h}(\Delta_\ell^h u(x)) \to \partial_{\ell\ell} u(x)$, and ψ^2 works as cut-off function, taking the square to have it non-negative and to simplify the estimates by Cauchy-Schwarz in the following.

[29] To simplify notation we keep the convention that repeated indices are summed over $1, \dots, n$.

Estimate for I_c. Similarly, using discrete integration by parts again,

$$\begin{aligned}|I_c| &= \left|\int_{\mathcal{U}} c\,u\,\Delta_\ell^{-h}(\psi^2\Delta_\ell^h u)\right| = \left|-\int_{\tilde{\mathcal{W}}} \Delta_\ell^h(cu)\,\psi^2\Delta_\ell^h u\right| \\ &= \left|-\int_{\tilde{\mathcal{W}}} \big[(\tau_{he_\ell}c)\,\Delta_\ell^h u + (\Delta_\ell^h c)\,u\big]\psi^2\Delta_\ell^h u\right| \\ &\le \int_{\tilde{\mathcal{W}}} \psi^2\big(|\tau_{he_\ell}c|\,|\Delta_\ell^h u|^2 + |\Delta_\ell^h c|\,|u|\,|\Delta_\ell^h u|\big) \\ &\le C\int_{\tilde{\mathcal{W}}} \psi^2|\Delta_\ell^h u|^2 + C\|u\|_{L^2(\tilde{\mathcal{W}})}\|\Delta_\ell^h u\|_{L^2(\tilde{\mathcal{W}})} \\ &\le C\|Du\|^2_{L^2(\tilde{\mathcal{W}})} + C\|u\|^2_{L^2(\tilde{\mathcal{W}})} \le C\|u\|^2_{H^1(\tilde{\mathcal{W}})},\end{aligned}$$

Estimate for I_f. We need to keep f undifferentiated. We estimate

$$\begin{aligned}|I_f| &= \left|\int_{\tilde{\mathcal{W}}} f\,\Delta_\ell^{-h}(\psi^2\Delta_\ell^h u)\right| \le \|f\|_{L^2(\tilde{\mathcal{W}})}\,\|\Delta_\ell^{-h}(\psi^2\Delta_\ell^h u)\|_{L^2(\tilde{\mathcal{W}})} \\ &\le C\|f\|_{L^2(\tilde{\mathcal{W}})}\,\|D(\psi^2\Delta_\ell^h u)\|_{L^2(\tilde{\mathcal{W}})} \\ &\le C\|f\|_{L^2(\tilde{\mathcal{W}})}\Big(\Big(\int_{\tilde{\mathcal{W}}} \psi^2|D\Delta_\ell^h u|^2\Big)^{1/2} + \|\Delta_\ell^h u\|_{L^2(\tilde{\mathcal{W}})}\Big) \\ &\le \varepsilon\int_{\tilde{\mathcal{W}}} \psi^2|D\Delta_\ell^h u|^2 + C_\varepsilon\|f\|^2_{L^2(\tilde{\mathcal{W}})} + C\|f\|_{L^2(\tilde{\mathcal{W}})}\|Du\|_{L^2(\tilde{\mathcal{W}})} \\ &\le \varepsilon\int_{\tilde{\mathcal{W}}} \psi^2|D\Delta_\ell^h u|^2 + C_\varepsilon\big(\|u\|^2_{H^1(\tilde{\mathcal{W}})} + \|f\|^2_{L^2(\tilde{\mathcal{W}})}\big),\end{aligned}$$

Estimate for I_a. Using discrete integration by parts in the ℓ−direction,

$$\begin{aligned}-I_a &= \int_{\tilde{\mathcal{W}}} \Delta_\ell^h(a_{ij}D_iu)\,D_j(\psi^2\Delta_\ell^h u) \\ &= \int_{\tilde{\mathcal{W}}} \big[(\tau_{he_\ell}a_{ij})\,\Delta_\ell^h D_iu + (\Delta_\ell^h a_{ij})\,D_iu\big]\big(\psi^2 D_j\Delta_\ell^h u + 2\psi(\partial_j\psi)\Delta_\ell^h u\big) \\ &= \int_{\tilde{\mathcal{W}}} \psi^2(\tau_{he_\ell}a_{ij})\,\Delta_\ell^h D_iu\,\Delta_\ell^h D_ju + 2\int_{\tilde{\mathcal{W}}} (\tau_{he_\ell}a_{ij})\,\Delta_\ell^h D_iu\,\Delta_\ell^h u\,\psi\partial_j\psi \\ &\quad + \int_{\tilde{\mathcal{W}}} \psi^2(\Delta_\ell^h a_{ij})\,D_iu\,\Delta_\ell^h D_ju + 2\int_{\tilde{\mathcal{W}}} (\Delta_\ell^h a_{ij})\,D_iu\,\Delta_\ell^h u\,\psi\partial_j\psi \\ &\ge \theta\int_{\tilde{\mathcal{W}}} \psi^2|\Delta_\ell^h Du|^2 - C\int_{\tilde{\mathcal{W}}} \psi^2|\Delta_\ell^h Du||Du| - C\int_{\tilde{\mathcal{W}}} |\Delta_\ell^h Du||\Delta_\ell^h u| - C\int_{\tilde{\mathcal{W}}} |Du||\Delta_\ell^h u| \\ &\ge \theta\int_{\tilde{\mathcal{W}}} \psi^2|\Delta_\ell^h Du|^2 - \varepsilon\int_{\tilde{\mathcal{W}}} \psi^2|D\Delta_\ell^h u|^2 - C_\varepsilon\|u\|^2_{H^1(\tilde{\mathcal{W}})},\end{aligned}$$

Conclusion. Since $-I_a = I_b + I_c - I_f$, combining the above bounds gives, for any $\varepsilon > 0$,

$$\theta\int_{\tilde{\mathcal{W}}} \psi^2|\Delta_\ell^h Du|^2 \le \varepsilon\int_{\tilde{\mathcal{W}}} \psi^2|D\Delta_\ell^h u|^2 + C_\varepsilon\Big(\|u\|^2_{H^1(\tilde{\mathcal{W}})} + \|f\|^2_{L^2(\tilde{\mathcal{W}})}\Big).$$

Choosing $\varepsilon > 0$ sufficiently small and absorbing the $\int \psi^2|D\Delta_\ell^h u|^2$ term into the left-hand side, we obtain

$$\int_{\tilde{\mathcal{W}}} \psi^2|\Delta_\ell^h Du|^2 \le C\Big(\|u\|^2_{H^1(\tilde{\mathcal{W}})} + \|f\|^2_{L^2(\tilde{\mathcal{W}})}\Big).$$

Since $\psi \equiv 1$ on $\mathcal{V}$, this implies

$$\|\Delta_\ell^h Du\|_{L^2(\mathcal{V})}^2 \le C\Big(\|u\|_{H^1(\tilde{\mathcal{W}})}^2 + \|f\|_{L^2(\tilde{\mathcal{W}})}^2\Big),$$

and letting $h \to 0$ we conclude

$$\|u\|_{H^2(\mathcal{V})}^2 \le C\Big(\|u\|_{H^1(\tilde{\mathcal{W}})}^2 + \|f\|_{L^2(\tilde{\mathcal{W}})}^2\Big).$$

Also, testing the equation against u, using uniform ellipticity, Cauchy-Schwarz and Young we get on $\tilde{W}$

$$\begin{aligned}\theta\|Du\|_{L^2}^2 &\le \int a_{ij}D_ju\,D_iu = \int fu - \int b_iD_iu\,u - \int cu^2\\ &\le \|f\|_{L^2}\|u\|_{L^2} + \|b\|_{L^\infty}\|Du\|_{L^2}\|u\|_{L^2} + \|c\|_{L^\infty}\|u\|_{L^2}^2\\ &\le \varepsilon\|Du\|_{L^2}^2 + C_\varepsilon\big(\|u\|_{L^2}^2 + \|f\|_{L^2}^2\big) \le \varepsilon\|Du\|_{L^2}^2 + C_\varepsilon\big(\|Du\|_{L^2}^2 + \|f\|_{L^2}^2\big),\end{aligned}$$

and we absorb the $\varepsilon\|Du\|_{L^2}^2$ term in the left-hand side. Combining this bound with the previous inequality gives the desired interior H^2 estimate. ☐

Remark 4.25.
1. One can also prove for operators with analytic coefficients, that solutions are real-analytic; in particular the interior Cauchy problem with Cauchy data on $\Sigma \subset \mathcal{U}$ that fail to be real-analytic has no solution.
2. This is a **local** result (away from the boundary). Thus **singularities do not propagate** in from the boundary or from rough regions of f; this non-propagation is characteristic of elliptic (and more generally hypoelliptic) equations. In general, one expects singularities to propagate along directions where the principal symbol vanishes.
3. Because of locality, the proof only needs uniform ellipticity on compact subsets of $\mathcal{U}$; degeneracy may occur near $\partial\mathcal{U}$.

We finally turn to regularity up to the boundary.

Theorem 4.26 (Boundary elliptic regularity, divergence form)**.** *Let $\mathcal{U} \subset \mathbb{R}^n$ be a bounded domain with $\partial\mathcal{U}$ of class C^k, $k \ge 2$. Let L be uniformly elliptic in divergence form, with $a_{ij} = a_{ji}$, $a_{ij} \in C^{k-1}(\overline{\mathcal{U}})$, and $b_i, c \in C^{k-2}(\overline{\mathcal{U}})$. If $f \in H^{k-2}(\mathcal{U})$ and $u \in H_0^1(\mathcal{U})$ is a weak solution of $Lu = f$ in $\mathcal{U}$, then*

$$\|u\|_{H^k(\mathcal{U})} \le C\big(\|u\|_{L^2(\mathcal{U})} + \|f\|_{H^{k-2}(\mathcal{U})}\big)$$

for some constant C depending on $\mathcal{U}$, k, n and the coefficients.

Proof. We argue as in the interior case. As in the interior case, the case $k > 2$ follows by induction from the case $k = 2$. For $k = 2$ it suffices to assume $a_{ij} \in C^1(\overline{\mathcal{U}})$ and $b_i, c \in L^\infty(\mathcal{U})$. By localization, flattening of $\partial\mathcal{U}$, and a partition of unity, it is enough to prove a local estimate near the boundary. Thus we may assume

$$\mathcal{U} = B(0,1) \cap \{x_n > 0\}, \qquad V := B(0,\tfrac{1}{2}) \cap \{x_n > 0\},$$

and choose $\psi \in C_c^\infty(B(0,1))$ with $\psi \equiv 1$ on V.

For $\ell = 1, \dots, n-1$ and $|h|$ small, consider the tangential difference quotients. Since e_ℓ is tangential to $\{x_n = 0\}$, for $|h|$ small these are well defined on $\operatorname{supp}\psi \subset \mathcal{U}$. Define the test function $v := -\Delta_\ell^{-h}(\psi^2 \Delta_\ell^h u)$ as in the interior case. For $|h|$ small, $v \in H_0^1(\mathcal{U})$ because $u \in H_0^1(\mathcal{U})$ and ψ has compact support in $B(0,1)$. The weak formulation

$$\int_{\mathcal{U}} a_{ij} D_j u \, D_i v + \int_{\mathcal{U}} b_i D_i u \, v + \int_{\mathcal{U}} cuv = \int_{\mathcal{U}} fv$$

holds for all $v \in H_0^1(\mathcal{U})$. Plugging in $v = -\Delta_\ell^{-h}(\psi^2 \Delta_\ell^h u)$ and using discrete integration by parts in the ℓ−direction, the product rules and the uniform ellipticity of (a_{ij}), one obtains, exactly as in the interior case but with only tangential difference quotients,

$$\int_V |\Delta_\ell^h Du|^2 \leq \int_{\mathcal{U}} \psi^2 |\Delta_\ell^h Du|^2 \leq C\Big(\|f\|^2_{L^2(\mathcal{U})} + \|u\|^2_{L^2(\mathcal{U})} + \|Du\|^2_{L^2(\mathcal{U})}\Big),$$

for all $|h|$ sufficiently small and all $\ell = 1, \dots, n-1$, with C independent of h. Letting $h \to 0$ and using the characterization of weak derivatives by difference quotients, we obtain

$$\int_V |D_\ell Du|^2 \leq C\Big(\|f\|^2_{L^2(\mathcal{U})} + \|u\|^2_{L^2(\mathcal{U})} + \|Du\|^2_{L^2(\mathcal{U})}\Big), \qquad \ell = 1, \dots, n-1.$$

Thus all $D^2_{ij} u$ with at least one tangential index ($i \leq n-1$ or $j \leq n-1$) belong to $L^2(V)$ with the same bound.

To control $D^2_{nn} u$ we use the equation. By interior regularity for divergence form operators, $u \in H^2_{\text{loc}}(\mathcal{U})$, so we may rewrite

$$-D_i(a_{ij} D_j u) + b_i D_i u + cu = f \quad \text{a.e. in } \mathcal{U}$$

and we can pass to the non-divergence form (here we use $a_{ij} \in C^1(\overline{\mathcal{U}})$ and the interior estimate $u \in H^2_{\text{loc}}(\mathcal{U})$, so that $D_i(a_{ij} D_j u) = (D_i a_{ij}) D_j u + a_{ij} D^2_{ij} u$ holds in the weak sense)

$$a_{ij} D^2_{ij} u + \tilde{b}_i D_i u + cu = f,$$

where $\tilde{b}_i := b_i + D_j a_{ij} \in L^\infty(\mathcal{U})$. Hence

$$a_{nn} D^2_{nn} u = -\sum_{(i,j) \neq (n,n)} a_{ij} D^2_{ij} u - \tilde{b}_i D_i u - cu + f.$$

Uniform ellipticity implies $a_{nn} \geq \theta > 0$ a.e. (take $\xi = e_n$ in the ellipticity inequality), so $1/a_{nn} \in L^\infty(V)$. All terms on the right-hand side belong to $L^2(V)$: the mixed second derivatives are already controlled, and $\tilde{b}_i, c \in L^\infty$, while $u, Du, f \in L^2(\mathcal{U})$. Therefore

$$D^2_{nn} u = \frac{1}{a_{nn}}\Big(-\sum_{(i,j) \neq (n,n)} a_{ij} D^2_{ij} u - \tilde{b}_i D_i u - cu + f\Big) \in L^2(V),$$

with

$$\|D^2_{nn} u\|_{L^2(V)} \leq C\Big(\|f\|_{L^2(\mathcal{U})} + \|u\|_{L^2(\mathcal{U})} + \|Du\|_{L^2(\mathcal{U})} + \sum_{(i,j) \neq (n,n)} \|D^2_{ij} u\|_{L^2(\mathcal{U})}\Big).$$

Combining the tangential estimate with this bound yields

$$\|u\|_{H^2(V)} \leq C\Big(\|u\|_{L^2(\mathcal{U})} + \|Du\|_{L^2(\mathcal{U})} + \|f\|_{L^2(\mathcal{U})}\Big).$$

thus, estimating $\|Du\|_{L^2(\mathcal{U})}$ by $\|u\|_{L^2(\mathcal{U})} + \|f\|_{L^2(\mathcal{U})}$, as we did in the interior case, we conclude

$$\|u\|_{H^2(V)} \leq C\Big(\|u\|_{L^2(\mathcal{U})} + \|f\|_{L^2(\mathcal{U})}\Big).$$

□

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Remark 4.27. There is also a "pointwise/Hölder regularity theory" (see the Elliptic PDE Lent course): **maximum principles** control u in the interior by its boundary values for classical $u \in C^2(\overline{\mathcal{U}})$, and **Schauder estimates** (1934) show that if $a, b, c, f \in C^{k,\gamma}(\overline{\mathcal{U}})$ then $u \in C^{k+2,\gamma}(\overline{\mathcal{U}})$ for $k \in \mathbb{N}$ and $\gamma \in (0,1)$.

This concludes the study of the Dirichlet problem. As compared to the Cauchy problem, one boundary condition has been dropped (that one on $\partial_N u$) and the Cauchy hypersurface $\partial\mathcal{U}$ now encloses the domain $\mathcal{U}$. Dropping instead the boundary condition on u and prescribing that on $\partial_N u$ results in the **Neumann problem**

$$\begin{cases} Lu = f & \text{in } \mathcal{U}, \\ \partial_N u = g & \text{on } \partial\mathcal{U}, \end{cases}$$

For the Neumann problem the functional setup and regularity theory are very similar to the Dirichlet case (one works in H^1 and obtains the same interior and boundary H^k- and $C^{k,\alpha}$-regularity under analogous assumptions on the coefficients and data), but the operator now has always a non-trivial kernel (constants). The existence requires a compatibility condition between f and g (e.g. $\int_{\mathcal{U}} f + \int_{\partial\mathcal{U}} g = 0$ in the model case $L = -\Delta$) and uniqueness holds only up to addition of constants. In particular, one cannot in general reduce a non-homogeneous Neumann condition to a homogeneous one as we do for Dirichlet problem.

5 Hyperbolicity

5.1 The notion of hyperbolicity

Very roughly, hyperbolic equations are those for which the Cauchy problem (prescribing initial data on a hypersurface) is the "right" notion of well-posedness in finite regularity. This is the broad class of PDEs for which some analogue of the Cauchy–Kovalevskaya theorem survives outside the analytic category.

Let L be a linear differential operator of order $k \geq 1$ on an open set $\mathcal{U} \subset \mathbb{R}^{n+1}$, with coordinates $y = (y_0, \ldots, y_n)$. We write

$$Lu = \sum_{|\alpha| \leq k} a_\alpha(y)\, \partial_y^\alpha u.$$

Recall that the *principal symbol* of L is the homogeneous polynomial of degree k

$$\sigma_p(L)(y, \eta) := \sum_{|\alpha| = k} a_\alpha(y)\, \eta^\alpha, \qquad \eta \in \mathbb{R}^{n+1}.$$

Also, a nonzero vector η is called *characteristic* at y if $\sigma_p(L)(y, \eta) = 0$. A smooth hypersurface $S \subset \mathcal{U}$ is *non-characteristic* at $y \in S$ if every nonzero normal vector to S at y is non-characteristic.

Definition 5.1. Let L be a linear differential operator of order $k \geq 1$ on an open set $\mathcal{U} \subset \mathbb{R}^{n+1}$. We say that L is (locally) *hyperbolic* at a point $y^0 \in \mathcal{U}$ if there exists a local coordinate system $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$ in a neighbourhood of y^0 such that, for every y in that neighbourhood and every $\xi \in \mathbb{R}^n \setminus \{0\}$, the map

$$\eta_t \longmapsto \sigma_p(L)(y, \eta_t, \xi), \qquad \eta = (\eta_t, \xi),$$

is a polynomial of degree k in η_t with k real roots (counted with multiplicity). Also, we say that L is (locally) *strictly hyperbolic* at y^0 if, in addition, for every y in that neighbourhood and every $\xi \in \mathbb{R}^n \setminus \{0\}$, these k real roots in η_t are simple (i.e. pairwise distinct).

Example 5.2 (First-order operators)**.** Consider a first-order operator

$$Lu(y) = \sum_{j=0}^{n} a_j(y)\,\partial_{y_j} u(y),$$

with real coefficients and $a(y^0) \neq 0$. Its principal symbol is $\sigma_p(L)(y,\eta) = \sum_{j=0}^{n} a_j(y)\,\eta_j$. We can choose new coordinates $y = (t,x)$ so that at y^0 the coefficient of ∂_t does not vanish, $a_t(y^0) \neq 0$ (if for some choice of t we have $a_t(y^0) = 0$, then the hypersurface $\{t = \mathrm{const}\}$ is characteristic there, and we change coordinates). In these coordinates,

$$\sigma_p(L)(y^0,\eta_t,\xi) = a_t(y^0)\,\eta_t + \sum_{j=1}^{n} a_j(y^0)\,\xi_j.$$

For each fixed $\xi \in \mathbb{R}^n$ this is a linear polynomial in η_t with the unique real root $\eta_t = -\frac{\sum_{j=1}^{n} a_j(y^0)\,\xi_j}{a_t(y^0)}$. Thus any first-order linear PDE with real coefficients is locally hyperbolic of order 1 at points where not all a_j vanish.

Example 5.3 (Second-order divergence form operators)**.** We now specialise to second-order scalar operators in divergence form

$$Lu(y) := \sum_{i=0}^{n} \partial_{y_i}\Big(\sum_{j=0}^{n} \tilde{a}_{ij}(y)\,\partial_{y_j} u(y)\Big) + \sum_{i=0}^{n} \tilde{b}_i(y)\,\partial_{y_i} u(y) + \tilde{c}(y)\,u(y) = \tilde{f}(y),$$

with smooth coefficients on $\mathbb{R}^{n+1}$. The principal symbol is the quadratic form

$$\sigma_p(L)(y,\eta) = \sum_{i,j=0}^{n} \tilde{a}_{ij}(y)\,\eta_i\eta_j, \qquad \eta \in \mathbb{R}^{n+1}.$$

Fix a point $y = y^0$. By an orthonormal change of variables we can diagonalise the constant real symmetric matrix $A(y^0) = (\tilde{a}_{ij}(y^0))_{ij}$, and assume that at y^0 the principal symbol is non-degenerate and has Lorentzian signature $(1,n)$ (that is, one eigenvalue is positive and n are negative, or vice versa). In other words,

$$\sigma_p(L)(y^0,\eta) = \lambda_{n+1}^2\eta_{n+1}^2 - \sum_{i=1}^{n} \lambda_i^2\eta_i^2, \qquad \lambda_i > 0.$$

Renaming $\eta_{n+1} = \eta_t$ and $\eta_i = \xi_i$ for $1 \le i \le n$, the characteristic equation

$$\lambda_{n+1}^2\eta_t^2 - \sum_{i=1}^{n} \lambda_i^2\xi_i^2 = 0$$

has the two real roots

$$\eta_t = \pm\frac{1}{\lambda_{n+1}}\Big(\sum_{i=1}^{n} \lambda_i^2\xi_i^2\Big)^{1/2}$$

for all $\xi \in \mathbb{R}^n$, so L is hyperbolic of order 2 at y^0 (indeed, strictly hyperbolic with respect to t in this case). If we restrict to operators of *wave type*, meaning that in the coordinates $y = (t, x)$ the mixed time–space second-order terms in the principal part vanish, i.e. $\tilde{a}_{0j}(y) = \tilde{a}_{j0}(y) = 0$ for all $j \geq 1$, after rescaling the corresponding coordinate and renaming variables as $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$, we can locally rewrite the equation in the form

$$\begin{aligned}\partial_{tt} u(t,x) = &\sum_{i=1}^n \partial_{x_i}\Big(\sum_{j=1}^n a_{ij}(t,x)\,\partial_{x_j} u(t,x)\Big) \\ &- \sum_{i=1}^n b_i(t,x)\,\partial_{x_i} u(t,x) - b(t,x)\,\partial_t u(t,x) - c(t,x)\,u(t,x) + f(t,x),\end{aligned} \tag{5.1}$$

where the matrix $(a_{ij}(t,x))_{1\leq i,j\leq n}$ is uniformly elliptic:

$$\sum_{i,j=1}^n a_{ij}(t,x)\,\xi_i\xi_j \;\geq\; \theta|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and some } \theta > 0.$$

The case $(a_{ij}) = \mathrm{Id}$, $b = 0$, $c = 0$, $f = 0$, corresponds to the **wave equation**

$$\Box u := \partial_{tt}^2 u - \Delta_x u = 0.$$

The set $\{(x,t) \,:\, t = 0\}$ is locally a non-characteristic hypersurface so we can hope to solve a Cauchy problem. In order to go beyond local existence, we will add additional boundary conditions: given a cylindrical domain $\mathcal{U} = (0,T) \times U$, we will specify $u_{|t=0}$ and $\partial_t u_{|t=0}$ (initial data), as well as boundary data in $x \in \partial U$ at each time, resulting in an **initial boundary value problem** (IBVP).

5.2 The IBVP for second-order PDEs

We begin with the situation that is closest to the one treated in the previous chapter.

5.2.1 The weak formulation

Let $U \subset \mathbb{R}^n$ be a bounded open set with $\partial U \in C^1$, and fix $T > 0$. Set

$$\mathcal{U}_T := (0,T) \times U,$$

and decompose its boundary as

$$\partial\mathcal{U}_T = \Sigma_0 \cup \Sigma_T \cup \partial^*\mathcal{U}_T,$$

where

$$\Sigma_0 := \{0\} \times U, \qquad \Sigma_T := \{T\} \times U, \qquad \partial^*\mathcal{U}_T := [0,T] \times \partial U.$$

We consider the initial boundary value problem

$$\begin{cases} \partial_{tt}^2 u + Lu = f & \text{in } \mathcal{U}_T, \\ u = \psi_0 & \text{on } \Sigma_0, \\ \partial_t u = \psi_1 & \text{on } \Sigma_0, \\ u = 0 & \text{on } \partial^*\mathcal{U}_T, \end{cases} \tag{5.2}$$

where $f = f(t,x)$, $\psi_0 = \psi_0(x)$ and $\psi_1 = \psi_1(x)$ are given data, and

$$Lu := -\partial_{x_i}\big(a_{ij}\partial_{x_j} u\big) + b_i\partial_{x_i} u + cu$$

with coefficient fields a_{ij}, b_i, c smooth on $\overline{\mathcal{U}_T}$. We assume that $a = (a_{ij})$ is symmetric and uniformly elliptic, i.e.

$$a_{ij}(t,x)\,\xi_i\xi_j \;\geq\; \theta|\xi|^2 \quad \text{for all } (t,x) \in \mathcal{U}_T,\ \xi \in \mathbb{R}^n,$$

for some constant $\theta > 0$.

As in the previous chapter, we look for a weak formulation that makes sense for $u \in H^1(\mathcal{U}_T)$. To derive it, start with a classical solution $u \in C^2(\overline{\mathcal{U}_T})$ of (5.2) and test the equation against a function

$$v \in C^2(\overline{\mathcal{U}_T}), \qquad v = 0 \text{ on } \partial^*\mathcal{U}_T \cup \Sigma_T.$$

We purposely do not impose $v = 0$ on Σ_0, because we want the boundary condition on $\partial_t u$ at $t = 0$ to appear in the weak formulation. Thus the boundary condition on $\partial^*\mathcal{U}_T$ is again encoded in the choice of the function space for the test functions.

Integrating by parts in t and x gives

$$\int_{\mathcal{U}_T} fv = \int_{\mathcal{U}_T} \Big[-(\partial_t u)(\partial_t v) + a_{ij}(\partial_{x_i}u)(\partial_{x_j}v) + b_i(\partial_{x_i}u)\,v + cuv \Big] - \int_{\Sigma_0} \psi_1 v. \tag{5.3}$$

For $u \in C^2(\overline{\mathcal{U}_T})$ the combination of (5.3) for all $v \in C^2(\overline{\mathcal{U}_T})$ with $v = 0$ on $\partial^*\mathcal{U}_T \cup \Sigma_T$, together with $u = \psi_0$ on Σ_0 and $u = 0$ on $\partial^*\mathcal{U}_T$, is equivalent to the original problem (5.2). Indeed:

- Testing (5.3) with $v \in C_c^\infty(\mathcal{U}_T)$ and integrating by parts recovers the PDE $\partial_{tt}^2 u + Lu = f$ in $\mathcal{U}_T$.
- Testing (5.3) with $v \in C^\infty(\mathcal{U}_T)$ that vanishes on $\partial^*\mathcal{U}_T \cup \Sigma_T$ but not necessarily on Σ_0 yields $\int_{\Sigma_0}(\psi_1 - \partial_t u)v = 0$ for all such v, hence $\partial_t u = \psi_1$ on Σ_0.

This motivates the following definition.

Definition 5.4 (Weak solutions to the second-order IBVP)**.** Let $T > 0$, $U \subset \mathbb{R}^n$ be bounded with $\partial U \in C^1$, and $\mathcal{U}_T = (0,T) \times U$. Assume

$$f \in L^2(\mathcal{U}_T), \qquad \psi_0 \in H_0^1(U), \qquad \psi_1 \in L^2(U),$$

and

$$a_{ij}, b_i, c \in L^\infty(\mathcal{U}_T).$$

A function $u \in H^1(\mathcal{U}_T)$ is called a weak solution of (5.2) if its traces satisfy

$$u_{|\Sigma_0} = \psi_0, \qquad u_{|\partial^*\mathcal{U}_T} = 0,$$

and (5.3) holds for every $v \in H^1(\mathcal{U}_T)$ whose trace vanishes on $\partial^*\mathcal{U}_T \cup \Sigma_T$.

Remark 5.5. The phrase “in the trace sense” means that we use the trace operator from Theorem 3.18

$$\mathrm{Tr} : H^1(\mathcal{U}_T) \longrightarrow L^2(\Sigma_0) \times L^2(\Sigma_T) \times L^2(\partial^*\mathcal{U}_T)$$

and interpret, for example, $u_{|\Sigma_0} = \psi_0$ as an equality in $L^2(\Sigma_0)$ between the trace of u and the given boundary datum ψ_0.

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5.2.2 The energy estimate

The starting point in the study of evolution equations in weak form is the identification of **a priori (energy) estimates**. "A priori" means that, before proving existence of solutions, we temporarily assume that a smooth solution exists and look for a norm on which an estimate can be propagated from the boundary data into the domain.

To introduce the idea, consider the wave equation

$$\Box u := \partial_{tt}^2 u - \Delta_x u = 0$$

on $\mathcal{U}_T = (0,T) \times U$, for a smooth solution u satisfying

$$u_{|t=0} = \psi_0, \qquad (\partial_t u)_{|t=0} = \psi_1 \quad \text{on } U, \qquad u_{|\partial^* \mathcal{U}_T} = 0$$

(which implies $(\partial_t u)_{|\partial^* \mathcal{U}_T} = 0$ as well). For $t \in (0,T)$, set $\mathcal{U}_t := (0,t) \times U$ and test the equation with $\partial_t u$. Integration by parts gives

$$\begin{aligned} 0 &= \int_{\mathcal{U}_t} \big[(\partial_{tt}^2 u)(\partial_t u) - (\Delta_x u)(\partial_t u)\big] = \frac{1}{2}\int_{\mathcal{U}_t} \partial_t \big[(\partial_t u)^2 + |\nabla_x u|^2\big] \\ &= \frac{1}{2}\int_{\Sigma_t} \big[(\partial_t u)^2 + |\nabla_x u|^2\big] - \frac{1}{2}\int_{\Sigma_0} \big[(\partial_t u)^2 + |\nabla_x u|^2\big]. \end{aligned} \tag{5.4}$$

Thus the **energy**

$$E[u](t) := \frac{1}{2}\int_{\Sigma_t} \big[(\partial_t u)^2 + |\nabla_x u|^2\big]$$

is conserved in time: $E[u]$ is the same for all $t \in (0,T)$. In particular, when there is enough regularity to perform this calculation, the energy identity immediately implies uniqueness. For example, for the wave equation, if $\psi_0 = \psi_1 = 0$, then

$$\frac{1}{2}\int_{\Sigma_0} \big[\psi_1^2 + |\nabla_x \psi_0|^2\big] = 0,$$

and (5.4) gives $E[u](t) = 0$ for all $t \in (0,T)$, hence $u \equiv 0$ in $\mathcal{U}_T$.

In the general case corresponding to (5.3), one obtains similarly the energy identity

$$E[u](t) - E[u](0) = \int_{\mathcal{U}_t} \Big[\frac{1}{2}(\partial_t a_{ij})(\partial_{x_i} u)(\partial_{x_j} u) - b_i(\partial_{x_i} u)(\partial_t u) - cu(\partial_t u) + f(\partial_t u)\Big], \tag{5.5}$$

with

$$E[u](t) := \frac{1}{2}\int_{\Sigma_t} \big[(\partial_t u)^2 + a_{ij}\partial_{x_i} u\, \partial_{x_j} u\big], \qquad E[u](0) = \frac{1}{2}\int_{\Sigma_0} \big[\psi_1^2 + a_{ij}(0,\cdot)\, \partial_{x_i}\psi_0\, \partial_{x_j}\psi_0\big].$$

By the uniform ellipticity of (a_{ij}), the spatial part of the energy controls $|\nabla_x u|$, so together with the $(\partial_t u)^2$ term we obtain

$$E[u](t) \ \geq C \int_{\Sigma_t} |\nabla_{t,x} u|^2.$$

Thus if the right-hand side of (5.5) can be estimated in terms of the prescribed data on $\mathcal{U}_T$, we obtain a uniform bound

$$\sup_{t \in (0,T)} E[u](t) \leq C' \big(\|\psi_0\|_{H^1(U)}^2 + \|\psi_1\|_{L^2(U)}^2 + \|f\|_{L^2(\mathcal{U}_T)}^2\big),$$

which yields $u \in L^\infty(0,T; H^1(U))$ and $\partial_t u \in L^\infty(0,T; L^2(U))$, where $L^\infty(0,T;X)$ is defined by the space of functions such that $\|u\|_{L^\infty(0,T;X)} := \operatorname{ess\,sup}_{t \in (0,T)} \|u(t)\|_X < \infty$.

5.2.3 Uniqueness

In getting uniqueness for the IBVP, we would like to use the energy estimates but for the moment the regularity is only $H^1(\mathcal{U}_T)$ and second derivatives need not exist. The main idea to overcome this difficulty is to *swap* the roles of u and the test function and to choose a test function v so that $\partial_t v$ behaves like u, thereby reducing the number of derivatives falling on u in the estimates.

Theorem 5.6 (Uniqueness of weak solutions). *Let $T > 0$ and $U \subset \mathbb{R}^n$ be bounded with $\partial U \in C^1$, and set $\mathcal{U}_T = (0,T) \times U$. Assume*

$$f \in L^2(\mathcal{U}_T), \qquad \psi_0 \in H_0^1(U), \qquad \psi_1 \in L^2(U),$$

and

$$a_{ij},\ \partial_t a_{ij},\ b_i,\ \nabla_x b_i,\ c \in L^\infty(\mathcal{U}_T).$$

Then there is at most one weak solution of (5.2) in the sense of Definition 5.4.

Proof of Theorem 5.6. Since the problem is linear, the statement is equivalent to proving that the only weak solution for zero data is the zero solution. We therefore consider $\psi_0 = \psi_1 = 0$ on U and $f = 0$ on $\mathcal{U}_T$. We then consider the test function $v(t,x) := \int_t^T u(s,x)e^{-\lambda s}\,\mathrm{d}s$ which is in $H^1(\mathcal{U}_T)$ since $u \in H^1(\mathcal{U}_T)$, and satisfies $v = 0$ on Σ_T by construction and $v = \partial_t v = 0$ on $\partial^* \mathcal{U}_T$ since $u = 0$ on $\partial^*\mathcal{U}_T$. Standard integration theorems show $\partial_t v = -ue^{-\lambda t}$ almost everywhere, and (5.3) yields

$$0 = \int_{\mathcal{U}_T} \left[(\partial_t u)ue^{-\lambda t} - a_{ij}(\partial^2_{ti}v)(\partial_j v)e^{\lambda t} + b_i(\partial_i u)v + cuv\right] =: A + B$$

with (using the symmetry $a_{ij} = a_{ji}$)

$$\begin{cases} A := \dfrac{1}{2}\displaystyle\int_{\mathcal{U}_T} \partial_t\left[u^2e^{-\lambda t} - a_{ij}(\partial_i v)(\partial_j v)e^{\lambda t} - v^2e^{\lambda t}\right] + \dfrac{\lambda}{2}\displaystyle\int_{\mathcal{U}_T}\left[u^2e^{-\lambda t} + a_{ij}(\partial_i v)(\partial_j v)e^{\lambda t} + v^2e^{\lambda t}\right] \\ B := \displaystyle\int_{\mathcal{U}_T}\left[b_i(\partial_i u)v + (c-1)uv + \dfrac{1}{2}(\partial_t a_{ij})(\partial_i v)(\partial_j v)e^{\lambda t}\right]. \end{cases}$$

We then bound A from below (using $v = \partial_i v = 0$ on Σ_T and $u = 0$ on Σ_0)

$$\begin{aligned} A &= \int_{\Sigma_T} \frac{u^2}{2}e^{-\lambda t} + \int_{\Sigma_0}\left[\frac{a_{ij}}{2}(\partial_i v)(\partial_j v) + \frac{v^2}{2}\right] + \frac{\lambda}{2}\int_{\mathcal{U}_T}\left[u^2e^{-\lambda t} + v^2e^{\lambda t} + a_{ij}(\partial_i v)(\partial_j v)e^{\lambda t}\right] \\ &\geq \frac{\lambda}{2}\int_{\mathcal{U}_T}\left[u^2e^{-\lambda t} + v^2e^{\lambda t} + \theta\left|\nabla_x v\right|^2 e^{\lambda t}\right] \end{aligned}$$

and then bound $|B|$ from above (using $u = v = 0$ on $\partial^*\mathcal{U}_T$)

$$\begin{aligned} |B| &= \left|\int_{\mathcal{U}_T}\left[-(\partial_i b_i)uv - b_i u(\partial_i v) + (c-1)uv + \frac{1}{2}(\partial_t a_{ij})(\partial_i v)(\partial_j v)e^{\lambda t}\right]\right| \\ &\leq \frac{\|\partial_i b_i\|_\infty + \max\{\theta^{-1},1\}\|b_i\|_\infty + |c-1| + \theta^{-1}\|\partial_t a\|_\infty}{2}\int_{\mathcal{U}_T}\left[u^2e^{-\lambda t} + v^2e^{\lambda t} + \theta\left|\nabla_x v\right|^2 e^{\lambda t}\right]. \end{aligned}$$

This implies, with $\lambda > \|\partial_i b_i\|_\infty + \max\{\theta^{-1},1\}\|b_i\|_\infty + |c-1| + \theta^{-1}\|\partial_t a\|_\infty$, that $u = 0$ on $\mathcal{U}_T$. □

Remark 5.7. The conceptual idea behind this argument is to "commute ∂_t^{-1}" with the PDE (at the cost of some lower order error terms). This argument is reminiscent of the **vector field method** (1985), widely used to study quasilinear wave equations such as the Einstein equations. Note that for the wave equation, one can simplify the argument by taking $\lambda = 0$ and using the boundary term in A (check it).

5.2.4 Existence

We turn to existence. The difficulty is to find an approximation path based on an already known result of existence for differential equations: we will use the **Galerkin method** (1915) which consists in projecting on well-chosen finite dimensional spaces, and using ODE theory to construct the approximate solutions. These finite-dimensional spaces are given by the enumeration of a Hilbert orthonormal basis that generalises the Fourier basis: the eigenfunctions of the Dirichlet problem of the Laplacian on U. The convergence of the scheme is based on the generalization of the energy estimate (5.4), with an additional difficulty: we have to provide a "discrete" version of (5.4) on the approximate solutions, uniformly in the approximation.

Theorem 5.8 (Existence of weak solutions)**.** *Given $T > 0$, $U \subset \mathbb{R}^n$ open bounded with $\partial U \in C^1$, $\mathcal{U}_T = (0,T) \times U$, $f \in L^2(\mathcal{U}_T)$, $\psi_0 \in H^1_0(U)$, $\psi_1 \in L^2(U)$ and $a, b, c \in C^0(\overline{\mathcal{U}}_T)$, there exists a weak solution $u \in H^1(\mathcal{U}_T)$ to (5.2). Moreover, there exists $C = C(U, T, a, b, c, n) > 0$ such that*

$$\|u\|_{H^1(\mathcal{U}_T)} \leq C \left(\|f\|_{L^2(\mathcal{U}_T)} + \|\psi_0\|_{H^1(U)} + \|\psi_1\|_{L^2(U)} \right).$$

Proof of Theorem 5.8. By a standard density argument it is enough to prove the theorem under the additional assumptions $\psi_0, \psi_1 \in C^\infty_c(U)$ and $f \in C^\infty_c(\mathcal{U}_T)$.

Step 1: Dirichlet eigenfunctions. Consider the Dirichlet eigenvalue problem

$$-\Delta\varphi = \lambda\varphi \quad \text{in } U \subset \mathbb{R}^n, \qquad \varphi_{|\partial U} = 0.$$

By Theorems 4.21 and 4.26 there exists a sequence of eigenpairs $(\varphi_k, \lambda_k)_{k\geq 1}$ with $\varphi_k \in H^1_0(U) \cap C^\infty(U)$ such that

$$0 < \lambda_1 < \lambda_2 < \dots, \qquad \lambda_k \to \infty \quad \text{as } k \to \infty.$$

The self-adjointness of $(\mu - \Delta)^{-1}$ (for any fixed $\mu > 0$ as in Theorem 4.21) implies that the eigenfunctions (φ_k) are orthogonal in $L^2(U)$. Moreover, if we multiply $-\Delta\varphi_k = \lambda_k\varphi_k$ by φ_ℓ and integrate by parts, we obtain orthogonality in $H^1_0(U)$ as well:

$$\int_U \nabla\varphi_k \cdot \nabla\varphi_\ell = \lambda_k \int_U \varphi_k\varphi_\ell = \lambda_k\delta_{k\ell}.$$

We may therefore choose $(\varphi_k)_{k\geq 1}$ orthonormal in $L^2(U)$ and orthogonal in $H^1_0(U)$.

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Step 2: Galerkin approximation. For $N \geq 1$ we look for

$$u_N(t,x) = \sum_{k=1}^{N} u_{N:k}(t)\,\varphi_k(x), \qquad t \in [0,T],\ x \in U,$$

with scalar coefficients $u_{N:k}(t)$. That is, we are looking at an ansatz that lives in a finite N-dimensional space. For an exact solution u one has $(u(0,\cdot),\varphi_k) = (\psi_0,\varphi_k)$ and $(\partial_t u(0,\cdot),\varphi_k) = (\psi_1,\varphi_k)$ for all k. We impose the same identities for the Galerkin approximation in the finite-dimensional space $\operatorname{span}\{\varphi_1,\dots,\varphi_N\}$, i.e.

$$u_{N:k}(0) = (\psi_0,\varphi_k), \qquad u'_{N:k}(0) = (\psi_1,\varphi_k), \quad k = 1,\dots,N.$$

We now impose that u_N satisfies the PDE in weak form, but only when tested against the finite-dimensional subspace $\operatorname{span}\{\varphi_1,\dots,\varphi_N\}$. More precisely, to choose how the coefficients $u_{N:k}(t)$ evolve, we insert the ansatz for u_N into (5.2) and, for each fixed $t \in [0,T]$ and each $k = 1,\dots,N$, we require that the weak formulation holds with test function φ_k:

$$\int_U \Big[\partial_{tt}u_N\,\varphi_k + a_{ij}(\partial_i u_N)(\partial_j\varphi_k) + b_i(\partial_i u_N)\varphi_k + b(\partial_t u_N)\varphi_k + cu_N\varphi_k\Big] = \int_U f\,\varphi_k. \tag{5.6}$$

Inserting $u_N(t,x) = \sum_{\ell=1}^{N} u_{N:\ell}(t)\varphi_\ell(x)$ and using the orthonormality of (φ_k) in $L^2(U)$, we obtain

$$u''_{N:k}(t) + \sum_{\ell=1}^{N} A_{N:k,\ell}(t)\,u_{N:\ell}(t) + \sum_{\ell=1}^{N} B_{N:k,\ell}(t)\,u'_{N:\ell}(t) = C_{N:k}(t),$$

for all $t \in [0,T]$ and $k = 1,\dots,N$, where

$$\begin{aligned}
A_{N:k,\ell}(t) &:= \int_{\Sigma_t} \Big(a_{ij}(t,x)\,\partial_i\varphi_\ell(x)\,\partial_j\varphi_k(x) + b_i(t,x)\,\partial_i\varphi_\ell(x)\,\varphi_k(x) + c(t,x)\,\varphi_\ell(x)\,\varphi_k(x)\Big)\,dx,\\
B_{N:k,\ell}(t) &:= \int_{\Sigma_t} b(t,x)\,\varphi_\ell(x)\,\varphi_k(x)\,dx,\\
C_{N:k}(t) &:= \int_{\Sigma_t} f(t,x)\,\varphi_k(x)\,dx.
\end{aligned}$$

This is a linear system of second-order ODEs with continuous coefficients on $[0,T]$, so (after rewriting it as a first-order system) it has a unique solution $(u_{N:k})_{k=1}^{N} \in C^2([0,T])$.

Step 3: A uniform $H^1(\mathcal{U}_T)$ estimate.

We now derive an energy estimate. We multiply (5.6) by $u'_{N:k}(t)e^{-\lambda t}$, integrate in $t \in [0,T'] \subset [0,T)$, and sum over k:

$$\int_{\mathcal{U}_{T'}} \Big[(\partial^2_{tt}u_N)(\partial_t u_N) + a_{ij}(\partial_i u_N)(\partial^2_{tj}u_N) + b_i(\partial_i u_N)(\partial_t u_N) + b(\partial_t u_N)^2 + cu_N(\partial_t u_N)\Big]e^{-\lambda s}\,dx\,ds = \int_{\mathcal{U}_{T'}} (\partial_t u_N) f e^{-\lambda s}\,dx\,ds.$$

By rearranging terms as in the proof of uniqueness, we obtain

$$
\begin{aligned}
\frac{1}{2}\int_{\mathcal{U}_{T'}} \partial_t\Big[\Big((\partial_t u_N(s))^2 + a_{ij}(s)(\partial_i u_N(s))(\partial_j u_N(s)) + (u_N(s))^2\Big)e^{-\lambda s}\Big]\,dx\,ds \\
+ \frac{\lambda}{2}\int_{\mathcal{U}_{T'}} \Big[(\partial_t u_N(s))^2 + a_{ij}(s)(\partial_i u_N(s))(\partial_j u_N(s)) + (u_N(s))^2\Big]e^{-\lambda s}\,dx\,ds \\
= \int_{\mathcal{U}_{T'}} \Bigg[\frac{1}{2}(\partial_t a_{ij}(s))(\partial_i u_N(s))(\partial_j u_N(s)) - b_i(s)(\partial_i u_N(s))(\partial_t u_N(s)) \\
- b(s)(\partial_t u_N(s))^2 - (c(s)-1)u_N(s)(\partial_t u_N(s)) + f(s)(\partial_t u_N(s))\Bigg]e^{-\lambda s}\,dx\,ds.
\end{aligned}
$$

The right-hand side is bounded (using Cauchy-Schwarz, Young's inequality and the boundedness of the coefficients) above by

$$
\mathrm{RHS} \le C\int_0^{T'} \Big(\|\partial_t u_N(s)\|^2_{L^2(U)} + \|\nabla_x u_N(s)\|^2_{L^2(U)} + \|u_N(s)\|^2_{L^2(U)} + \|f(s)\|^2_{L^2(U)}\Big)\,e^{-\lambda s}\,ds
$$

for some $C > 0$ independent of N. On the other hand, using the uniform ellipticity of (a_{ij}), the left-hand side is bounded below by

$$
\begin{aligned}
\mathrm{LHS} \ge\; & E[u_N](T')e^{-\lambda T'} - E[u_N](0) \\
& + \frac{\lambda}{2}\int_0^{T'} \Big(\|\partial_t u_N(s)\|^2_{L^2(U)} + \theta\|\nabla_x u_N(s)\|^2_{L^2(U)} + \|u_N(s)\|^2_{L^2(U)}\Big)\,e^{-\lambda s}\,ds,
\end{aligned}
$$

where

$$
E[u_N](t) := \frac{1}{2}\int_{\Sigma_t} \Big[(\partial_t u_N)^2 + a_{ij}(\partial_i u_N)(\partial_j u_N) + (u_N)^2\Big]\,dx \ge 0.
$$

For λ large enough we deduce, absorbing the exponential factors,

$$
E[u_N](T') + \|u_N\|^2_{H^1_{t,x}(\mathcal{U}_{T'})} \le C'\Big(E[u_N](0) + \|f\|^2_{L^2(\mathcal{U}_{T'})}\Big) \tag{5.7}
$$

for some constant $C' > 0$ independent of N. Letting $T' \to T$ we obtain

$$
\|u_N\|^2_{H^1_{t,x}(\mathcal{U}_T)} \le C'\Big(E[u_N](0) + \|f\|^2_{L^2(\mathcal{U}_T)}\Big)
$$

with a constant uniform in N.

We now estimate the initial energy. Denote

$$
\psi_0^N := \sum_{k=1}^N (\psi_0, \varphi_k)_{L^2(U)}\varphi_k, \qquad \psi_1^N := \sum_{k=1}^N (\psi_1, \varphi_k)_{L^2(U)}\varphi_k.
$$

By Bessel's inequality in $L^2(U)$,

$$
\|\psi_0^N\|_{L^2(U)} \le \|\psi_0\|_{L^2(U)}, \qquad \|\psi_1^N\|_{L^2(U)} \le \|\psi_1\|_{L^2(U)}.
$$

Moreover, using the orthogonality of (φ_k) in $H^1_0(U)$ and the relation $\|\nabla_x\varphi_k\|^2_{L^2(U)} = \lambda_k$ under our normalisation $\|\varphi_k\|_{L^2(U)} = 1$, we get

$$
\begin{aligned}
\|\nabla_x\psi_0^N\|^2_{L^2(U)} &= \sum_{k=1}^N (\psi_0, \varphi_k)^2_{L^2(U)}\|\nabla_x\varphi_k\|^2_{L^2(U)} = \sum_{k=1}^N \lambda_k(\psi_0, \varphi_k)^2_{L^2(U)} \\
&= \sum_{k=1}^N \left(\nabla_x\psi_0, \frac{\nabla_x\varphi_k}{\sqrt{\lambda_k}}\right)^2_{L^2(U)} \le \|\nabla_x\psi_0\|^2_{L^2(U)},
\end{aligned}
$$

since the sequence $(\nabla_x \varphi_k/\sqrt{\lambda_k})$ is orthonormal in $L^2(U)$. Hence

$$E[u_N](0) \le C'' \left(\|\psi_0\|^2_{H^1(U)} + \|\psi_1\|^2_{L^2(U)} \right)$$

uniformly in N, and we obtain

$$\|u_N\|^2_{H^1(\mathcal{U}_T)} \le C''' \left(\|\psi_0\|^2_{H^1(U)} + \|\psi_1\|^2_{L^2(U)} + \|f\|^2_{L^2(\mathcal{U}_T)} \right)$$

with a constant independent of N (recall that $H^1(\mathcal{U}_T)$ refers to gradients in both t and x).

Step 4: Compactness and weak derivatives. By the Rellich–Kondrachov theorem, there exists a subsequence (still denoted u_N) converging in $L^2(\mathcal{U}_T)$ to some $u \in L^2(\mathcal{U}_T)$, and at the same time bounded in $H^1(\mathcal{U}_T)$. Thus, up to extracting a subsequence once more, we may assume

$$u_N \rightharpoonup u \quad \text{weakly in } H^1(\mathcal{U}_T), \qquad u_N \to u \quad \text{strongly in } L^2(\mathcal{U}_T).$$

Here and below, $u_N \rightharpoonup u$ weakly in H (Hilbert space) means that

$$(u_N, \phi)_H \to (u, \phi)_H \quad \text{for every } \phi \in H,$$

and we used (Banach-Alaoglu theorem) that bounded sets in Hilbert spaces are weakly relatively compact.

Using our approximation results in Proposition 3.14 on the test functions, one checks that the derivatives $\partial_t u_N$ and $\partial_i u_N$ converge *weakly* to generalised derivatives $D_t u, D_i u \in L^2(\mathcal{U}_T)$, i.e.

$$(\partial_t u_N, \varphi)_{L^2(\mathcal{U}_T)} \to (D_t u, \varphi)_{L^2(\mathcal{U}_T)}, \qquad (\partial_i u_N, \varphi)_{L^2(\mathcal{U}_T)} \to (D_i u, \varphi)_{L^2(\mathcal{U}_T)}$$

for any given $\varphi \in L^2(\mathcal{U}_T)$. Thus $u \in H^1(\mathcal{U}_T)$ and $D_t u, D_i u$ are its weak derivatives.

Step 5: Passage to the limit in the weak formulation. Testing (5.6) against any

$$v_m(t,x) = \sum_{k=1}^{m} v_{m:k}(t) \varphi_k(x), \qquad m \le N,$$

with $v_{m:k} \in C^1([0,T])$ and $v_{m:k}(T) = 0$, and integrating over $(0,T)$, we obtain, by integration by parts in time,

$$\int_{\mathcal{U}_T} f\, v_m = \int_{\mathcal{U}_T} \Big(-\partial_t u_N\, \partial_t v_m + a_{ij} \partial_i u_N\, \partial_j v_m + b_i \partial_i u_N\, v_m + b\, \partial_t u_N\, v_m + c\, u_N\, v_m \Big) - \int_{\Sigma_0} \psi_1^N v_m.$$

Using the weak convergence $u_N \rightharpoonup u$ in $H^1(\mathcal{U}_T)$ and $\psi_1^N \to \psi_1$ in $L^2(U)$, we deduce

$$\int_{\mathcal{U}_T} f\, v_m = \int_{\mathcal{U}_T} \Big(-D_t u\, \partial_t v_m + a_{ij} D_i u\, \partial_j v_m + b_i D_i u\, v_m + b\, D_t u\, v_m + c\, u\, v_m \Big) - \int_{\Sigma_0} \psi_1 v_m.$$

We now identify the class of admissible test functions. The functions v_m of the above form are dense in

$$\{ v \in H^1(\mathcal{U}_T) \,:\, v_{|\Sigma_T \cup \partial^* \mathcal{U}_T} = 0 \},$$

where $\partial^* \mathcal{U}_T := (0,T) \times \partial U$. Indeed, the density follows from the density of the linear span of (φ_k) in $H^1_0(U)$.

We prove this last fact. Let V be the closed linear span of $\{\varphi_k\}_{k \ge 1}$ in $H^1_0(U)$ and suppose, for contradiction, that $V \ne H^1_0(U)$. Then $E := V^\perp \subset H^1_0(U)$ is a non-trivial closed subspace.

Because $(\mu - \Delta)^{-1}$ is self-adjoint and compact on $H_0^1(U)$, it leaves V and E invariant (one checks that $(\mu - \Delta)^{-1}(V) \subset V$, which implies the same for the orthogonal complement E), and its restriction

$$K := (\mu - \Delta)^{-1}_{|E} : E \to E$$

is again compact, self-adjoint and non-zero. A basic spectral property of compact self-adjoint operators states that any non-zero compact self-adjoint operator K on a Hilbert space has at least one non-zero eigenvalue. This contradicts the fact that, by construction, K has no eigenvalues on E. Hence $V = H_0^1(U)$, i.e. (φ_k) is dense in $H_0^1(U)$.

Thus, given any such v, there exists a sequence (v_m) with $v_m \to v$ in $H^1(\mathcal{U}_T)$. Since the trace operator is continuous from $H^1(\mathcal{U}_T)$ to $L^2(\Sigma_0)$, this implies $v_m|_{\Sigma_0} \to v|_{\Sigma_0}$ in $L^2(\Sigma_0)$ and therefore

$$\int_{\Sigma_0} \psi_1 v_m \to \int_{\Sigma_0} \psi_1 v.$$

Passing to the limit in the identity above then yields exactly the weak formulation (5.3) for all $v \in H^1(\mathcal{U}_T)$ with $v_{|\Sigma_T \cup \partial^* \mathcal{U}_T} = 0$.

Step 6: Initial and boundary conditions. We already know that

$$u_N(0,x) = \sum_{k=1}^{N} (\psi_0, \varphi_k)_{L^2(U)} \varphi_k(x) =: \psi_0^N(x).$$

Since (φ_k) is an orthonormal basis of $L^2(U)$, Parseval's identity gives $\psi_0^N \to \psi_0$ in $L^2(U)$. Moreover, using $-\Delta \varphi_k = \lambda_k \varphi_k$ and the orthonormality of $(\nabla \varphi_k / \sqrt{\lambda_k})$ in $L^2(U)$, one checks that $\psi_0^N \to \psi_0$ in $H_0^1(U)$. Since the trace operator $H^1(\mathcal{U}_T) \to L^2(U)$ is continuous, the weak convergence $u_N \rightharpoonup u$ in $H^1(\mathcal{U}_T)$ implies $u_N(0,\cdot) \rightharpoonup u(0,\cdot)$ in $L^2(U)$. On the other hand, $u_N(0,\cdot) = \psi_0^N \to \psi_0$ strongly (hence weakly) in $L^2(U)$, so the weak limit must be $u(0,\cdot) = \psi_0$.

For the boundary condition, note that each u_N vanishes on $\partial^* \mathcal{U}_T$. The set

$$V := \{ w \in H^1(\mathcal{U}_T) : w_{|\partial^* \mathcal{U}_T} = 0 \}$$

is a closed linear subspace of $H^1(\mathcal{U}_T)$, hence weakly closed. Since $u_N \in V$ for all N and $u_N \rightharpoonup u$ in $H^1(\mathcal{U}_T)$, the limit u also belongs to V, i.e. $u_{|\partial^* \mathcal{U}_T} = 0$. This completes the proof. □

Remark 5.9. In fact the energy estimate (5.7) established in the proof shows, by keeping the term $E[u_N](T')$, $T' \in (0,T)$, and taking the limit $N \to \infty$, that

$$E[u](T') + \|u\|^2_{H^1_{t,x}(\mathcal{U}_{T'})} \leq C(E[u](0) + \|f\|^2_{L^2(\mathcal{U}_{T'})}),$$

so that the solution constructed in the theorem satisfies $u \in L^\infty(0,T; H^1(U))$ and $\partial_t u \in L^\infty(0,T; L^2(U))$. Note that although the energy $E[u](t)$ on each time slice is bounded, it is not always continuous for such weak solutions merely in $H^1(\mathcal{U}_T)$.

5.2.5 Hyperbolic regularity theory

Just like for elliptic PDEs, we want to prove that the weak solutions we have constructed are more regular and in fact classical (strong) solutions when the coefficients and data are regular enough. Once more the core idea is best explained on the wave equation $\Box u = \partial^2_{tt} u - \Delta_x u = f$ in $\mathcal{U}_T$ with f, ψ_0, ψ_1 smooth. The previous theorems 5.6-5.8 show there a unique solution $u \in H^1(\mathcal{U}_T)$ with the boundary conditions $u_{|\Sigma_0} = \psi_0$, $(\partial_t u)_{|\Sigma_0} = \psi_1$ and $u_{|\partial^* \mathcal{U}_T} = 0$. Let us argue **a priori**, i.e. we assume that u is smooth and we try to establish an estimate on higher-order derivatives. Then $w := \partial_t u$ satisfies $\Box w = \partial_t f$ in $\mathcal{U}_T$ with the boundary conditions $w_{|\Sigma_0} = (\partial_t u)_{|\Sigma_0} = \psi_1$, $(\partial_t w)_{|\Sigma_0} = (\partial^2_{tt} u)_{|\Sigma_0} = (\Delta_x u)_{|\Sigma_0} + (\partial_t f)_{|\Sigma_0} = \Delta_x \psi_0 + (\partial_t f)_{|\Sigma_0}$ and $w_{|\partial^* \mathcal{U}_T} = 0$. Then the same energy estimate that we used in the last proof (of existence) shows $E[w](T') + \|w\|^2_{H^1_{t,x}(\mathcal{U}_{T'})} \lesssim E[w](0) + \|\partial_t f\|^2_{L^2(\mathcal{U}_{T'})}$ for $T' \in (0, T)$, which proves that $\partial_t u = w \in H^1_{t,x}(\mathcal{U}_T)$. Meanwhile $\Delta_x u = \partial_t u - f \in L^2(\mathcal{U}_T)$ implies by the basic ellipticity estimate that $\partial^2_{x_i x_j} u \in L^2(\mathcal{U}_T)$ for all i, j. So finally $u \in H^2(\mathcal{U}_T)$, and by induction we can bound similarly higher-order derivatives.

Theorem 5.10 (Hyperbolic regularity). *Let $T > 0$ and $k \geq 2$ be an integer, and let $U \subset \mathbb{R}^n$ be open, bounded, with $\partial U \in C^k$. Set $\mathcal{U}_T = (0, T) \times U$.*

Assume $a, b, c \in C^k(\overline{\mathcal{U}_T})$ and

$$\partial_t^\ell f \in L^\infty\big(0, T; H^{k-1-\ell}(U)\big) \qquad \text{for all } \ell = 0, \ldots, k-1.$$

Let $u \in H^1(\mathcal{U}_T)$ be the (unique) weak solution to (5.2) given by Theorem 5.8, with initial data $u(0, \cdot) = \psi_0$, and $\partial_t u(0, \cdot) = \psi_1$. Suppose in addition that the time traces of u at $t = 0$ satisfy

$$\partial_t^\ell u(0, \cdot) \in H^1_0(U) \quad \text{for } \ell = 0, \ldots, k-1, \qquad \partial_t^k u(0, \cdot) \in L^2(U).$$

Then $\partial_t^\ell u \in L^\infty\big(0, T; H^{k-\ell}(U)\big)$ for every $\ell = 0, \ldots, k$ (in particular $u \in H^k(\mathcal{U}_T)$).

LECTURE 23

Proof of Theorem 5.10. Step 1: Reduction to the case $k = 2$. Under our assumptions on L and U, the eigenfunctions $\varphi_k \in H_0^1(U)$ of the Dirichlet problem

$$L\varphi_k = \lambda_k \varphi_k \quad \text{in } U, \qquad \varphi_k = 0 \quad \text{on } \partial U,$$

are in $H^k(U)$ by Theorems 4.23–4.26. For general $k \geq 2$, differentiating the equation

$$\partial_{tt} u + Lu = f$$

$k-2$ times in t shows that each $\partial_t^m u$ ($m \leq k-2$) solves a hyperbolic equation of the same type, with coefficients still in C^{k-m} and right-hand side involving $\partial_t^\ell f$ and lower-order derivatives of u, which satisfy the same form of assumptions. If we know the theorem for $k = 2$, we can apply it successively to $\partial_t^m u$ for $m = 0, \dots, k-2$ and obtain the full statement. Hence it is enough to prove the case $k = 2$.

Step 2: Galerkin approximation and time regularity. In the Galerkin scheme of the previous proof, we write

$$u_N(t,x) = \sum_{k=1}^{N} u_{N:k}(t)\, \varphi_k(x),$$

and the coefficients $u_{N:k}$ solve a linear system of second-order ODEs

$$\ddot{u}_{N:k}(t) + \sum_{\ell=1}^{N} A_{N:k\ell}(t) u_{N:\ell}(t) + \sum_{\ell=1}^{N} B_{N:k\ell}(t) \dot{u}_{N:\ell}(t) = C_{N:k}(t).$$

Because $a, b, c \in C^2(\overline{\mathcal{U}_T})$ and f has the assumed time-regularity, the coefficients $A_{N:k\ell}(t), B_{N:k\ell}(t)$ are C^2 and $C_{N:k}(t)$ is C^1 in t. Standard ODE theory then implies $(u_{N:k}(t))_{k=1}^N \in C^3$ in t. We can therefore differentiate this system in t; denoting $v_N := \partial_t u_N$, the differentiated system is the Galerkin approximation for v_N, which solves the linearised equation for $\partial_t u$ (when projected against φ_k from $k = 1, \dots, N$).

Step 3: Energy estimate for $\partial_t u_N$. We now perform the energy estimate on the differentiated system. Testing the equation for v_N by $\partial_{tt}^2 u_{N:k}(t)\, e^{-\lambda t}$, integrating over $(0, T') \times U$ for $T' \in (0, T)$, and summing over $k = 1, \dots, N$, we obtain

$$E[\partial_t u_N](T') + \|\partial_t u_N\|^2_{H^1_{t,x}(\mathcal{U}_{T'})} \leq C \left(E[\partial_t u_N](0) + E[u_N](0) + \|f\|^2_{H^1(\mathcal{U}_{T'})} + \|\partial_t f\|^2_{L^2(\mathcal{U}_{T'})} \right)$$

for some constant $C > 0$ independent of N.

Step 4: Initial energies and elliptic regularity. At $t = 0$, the functions $u_N(0)$, $\partial_t u_N(0)$ and $\partial_{tt} u_N(0)$ are the L^2-orthogonal projections of ψ_0, ψ_1, ψ_2 onto $\operatorname{span}\{\varphi_1, \dots, \varphi_N\}$. Therefore

$$\|u_N(0)\|_{H^1(U)} \leq \|\psi_0\|_{H^1(U)}, \quad \|\partial_t u_N(0)\|_{H^1(U)} \leq \|\psi_1\|_{H^1(U)}, \quad \|\partial_{tt} u_N(0)\|_{L^2(U)} \leq \|\psi_2\|_{L^2(U)},$$

so $E[\partial_t u_N](0)$ and $E[u_N](0)$ are bounded uniformly in N. Letting $T' \to T$ in the estimate of Step 3, we obtain uniform bounds

$$\partial_{tt}^2 u_N,\ \partial_{ti}^2 u_N \in L^2(\mathcal{U}_T) \quad (i = 1, \dots, n).$$

From the equation $\partial_{tt}^2 u_N + L u_N = f$ it follows that $a_{ij} \partial_{ij}^2 u_N \in L^2(\mathcal{U}_T)$ uniformly in N. Since L is uniformly elliptic in x with C^2 coefficients and $u_N(t, \cdot) \in H_0^1(U)$, we can apply for almost every t Theorems 4.23–4.26 to $L(t,\cdot)u_N(t,\cdot) = f(t,\cdot) - \partial_t u_N(t,\cdot) \in L^2(U)$ with the Dirichlet boundary condition, thus we get

$$u_N(t,\cdot) \in H^2(U),$$

with an estimate independent of N and t. Integrating in time, we see that (u_N) is bounded in $H^2(\mathcal{U}_T)$ uniformly in N, and hence $u \in H^2(\mathcal{U}_T)$ in the limit $N \to \infty$.

Step 5: L^∞-in-time bounds and passage to the limit. We know that $u \in L^2(0,T;H^2(U))$ and we want to promote it to $u \in L^\infty(0,T;H^2(U))$. The energy estimates for u_N and $\partial_t u_N$ also give, for a constant C independent of N,

$$\|u_N\|_{L^\infty(0,T;H^2(U))} + \|\partial_t u_N\|_{L^\infty(0,T;H^1(U))} + \|\partial^2_{tt} u_N\|_{L^\infty(0,T;L^2(U))} \le C.$$

By weak compactness, up to a subsequence we have

$$u_N \rightharpoonup u \quad \text{in } L^2(0,T;H^2(U)), \quad \partial_t u_N \rightharpoonup \partial_t u \quad \text{in } L^2(0,T;H^1(U)),$$

and similarly for $\partial_{tt} u_N$ in $L^2(0,T;L^2(U))$. Moreover, for a.e. t we have $u_N(t) \rightharpoonup u(t)$ in $H^2(U)$, so by weak lower semicontinuity

$$\|u(t)\|^2_{H^2(U)} \le \liminf_{N\to\infty} \|u_N(t)\|^2_{H^2(U)}.$$

If we set $g_N(t) := \|u_N(t)\|^2_{H^2(U)}$, the uniform energy bound gives $g_N \in L^\infty(0,T)$ with $\|g_N\|_{L^\infty} \le C^2$, and Fatou's lemma yields

$$\int_0^T \|u(t)\|^2_{H^2(U)}\,dt \le \int_0^T \liminf_{N\to\infty} g_N(t)\,dt \le \liminf_{N\to\infty} \int_0^T g_N(t)\,dt \le TC^2.$$

Thus u inherits the same type of bound as the u_N; the same argument applies to $\partial_t u$ and $\partial_{tt} u$. Together with the pointwise-in-time energy estimate and weak lower semicontinuity of the energy, this shows that

$$u \in L^\infty(0,T;H^2(U)), \quad \partial_t u \in L^\infty(0,T;H^1(U)), \quad \partial^2_{tt} u \in L^\infty(0,T;L^2(U)).$$

□

Remark 5.11. 1. In particular, the theorem is a *propagation of regularity* result: for $k = 2$ it says that if the initial data are already H^2 (in fact the compatibility condition $\psi_2 + L\psi_0 = f(0,\cdot)$ in $L^2(U)$ forces $\psi_0 \in H^2(U)$ by elliptic regularity), then the solution remains H^2; there is no new smoothing coming from the time evolution. This lack of smoothing is already clear in the 1D wave equation on $\mathbb{R}$ (for simplicity, but the same is true on bounded intervals using Fourier expansion),

$$\partial_{tt} u - \partial_{xx} u = 0, \qquad u(0,x) = u_0(x),\ \partial_t u(0,x) = 0.$$

By d'Alembert's formula (see Exercise 4.12) we have

$$u(t,x) = \tfrac{1}{2} u_0(x+t) + \tfrac{1}{2} u_0(x-t),$$

so the spatial regularity of $u(t,\cdot)$ is exactly the same as that of u_0, for all t. For instance, if $u_0 \in H^1(\mathbb{R}) \setminus H^2(\mathbb{R})$, then $u(t,\cdot) \in H^1(\mathbb{R}) \setminus H^2(\mathbb{R})$ for every t.
With the same example but with $f \not\equiv 0$ and $u_0 \equiv 0$ one can check that u gains one space derivative with respect to f and no gain in t (which is what the statement of our theorem tells us in this regard).

2. To make a connection again with the **vector fields vector**, in the proof of uniqueness we have "almost commuted" the equation (in weak form) with ∂_t^{-1} (with an exponential weight in time), in the proof of existence we have "almost commuted" the equation (in discretised form) with ∂_t (with an exponential weight in time), and in this higher-order regularity estimate we have "almost commuted" the equation (in discretised form) with ∂^2_{tt} (with an exponential weight in time) and then used the PDE itself to recover the other higher-order derivatives.
3. When $k = 2$ we obtain the PDE as an equality almost everywhere. When $k > 2 + n/2$, by Sobolev inequalities, we obtain the PDE in the classical sense everywhere (check it). When $k = \infty$, we obtain that the solution u is smooth.
4. The case $k = 1$ corresponds to the previous theorem of existence for which the energy $E[u](t)$ is bounded on each time slice but not necessearily continuous as a function of t. However as soon as $k \geq 2$, the bound $\partial_t u \in L^\infty(0, T; H^1(U))$ implies $u \in C^0(0, T; H^1(U))$ (check it). When the latter continuity is true we say that u belongs to the **energy class** (that is the class of solutions for which the energy is well-defined and continuous in time).

5.2.6 Domain of dependence

A crucial feature of hyperbolic equations is the **finite speed of propagation of information**. For instance, sound waves travel at a certain maximal speed, depending on the medium, earthquakes waves travel at a certain maximal speeds, depending on the medium and different for the longitudinal and transversal waves, and electromagnetic waves travel at the speed of light. When considering the hyperbolic PDEs of general relativity, the feature implies the famous principle that "nothing can travel faster than the speed of light". Let us translate all this into a precise mathematical estimate. We need a concept of hypersurface never pointing to a direction that would entail a speed faster than allowed by the equation.

Definition 5.12. Given $V \subset \mathbb{R}^n$ open bounded and $\tau \in C^\infty(\overline{V})$ so that $\tau_{|V} > 0$ and $\tau_{|\partial V} = 0$, we denote the graph of $t = \tau(x)$ by $S := \{(\tau(x), x) \, : \, x \in V\} \subset \mathbb{R}^{n+1}$ hypersurface and the enclosed domain $\mathcal{D} := \{(t, x) \, : \, x \in V, \, t \in (0, \tau(x))\}$. We then say that S is **spacelike** for the hyperbolic equation $\partial^2_{tt} u + Lu = f$ defined in (5.2) if

$$a_{ij}(x)\partial_{x_i}\tau(x)\partial_{x_j}\tau(x) < 1$$

for all $x \in V$, and if so we say that $\mathcal{D}$ is a **domain of dependence** for V.

Theorem 5.13. *Given $T > 0$, $U \subset \mathbb{R}^n$ open bounded with $\partial U \in C^1$, $\mathcal{U}_T = (0, T) \times U$, $f \in H^1(\mathcal{U}_T)$, $\psi_0 \in H^1(U)$, $\psi_1 \in L^2(U)$, $a, \partial_t a, b, \nabla_x b, c \in L^\infty(\mathcal{U}_T)$, $u \in H^1(\mathcal{U}_T)$ a weak solution to (5.2) according to Definition 5.4, and $V \subset U$ open bounded with $\partial V \in C^1$ and $\tau \in C^\infty(\overline{V})$ so that $\tau(V) \subset (0, T)$ and $\tau_{|\partial V} = 0$ with $S := \{(\tau(x), x) \, : \, x \in V\} \subset \mathbb{R}^{n+1}$ spacelike, and $\mathcal{D} := \{(t, x) \, : \, x \in V, \, t \in (0, \tau(x))\}$.*
Then $u_{|\mathcal{D}}$ only depends on $(\psi_0)_{|V}$, $(\psi_1)_{|V}$ and $f_{|V}$: in particular if ψ_0, ψ_1 and f are vanishing in V, then $u = 0$ in $\mathcal{D}$.

Proof of Theorem 5.13. The proof is similar to that of uniqueness of weak solution (Theorem 5.6), except that we slightly modify the test function v to make it vanishes outside $\mathcal{D}$. Indeed by linearity, it is enough to prove that $u_{|\mathcal{D}} = 0$ as soon as $(\psi_0)_{|V} = (\psi_1)_{|V} = f_{|V} = 0$ (we have taken f in $H^1(\mathcal{U}_T)$ in order to be able to define its trace at $t = 0$). We then define

$v(t,x) := \int_t^{\tau(x)} u(s,x)e^{-\lambda s}\,\mathrm{d}s$ for $(t,x) \in \mathcal{D}$ and $v(t,x) = 0$ in $\mathcal{U}_T \setminus \mathcal{D}$. We then have $\partial_t v = -ue^{-\lambda t}$ for $(t,x) \in \mathcal{D}$ and $v = 0$ elsewhere, and $v \in H^1(\mathcal{U}_T)$ with $v = 0$ on $\Sigma_T \cup \partial^*\mathcal{U}_T$; moreover $v = 0$ on Σ_0 since the data are zero on V and on S, so in fact $v \in H_0^1(\mathcal{D})$. Performing the same estimate as in Theorem 5.6 we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{D}} \partial_t \left[u^2 e^{-\lambda t} - a_{ij}(\partial_i v)(\partial_j v)e^{\lambda t} - v^2 e^{\lambda t}\right] + \frac{\lambda}{2}\int_{\mathcal{D}} \left[u^2 e^{-\lambda t} + a_{ij}(\partial_i v)(\partial_j v)e^{\lambda t} + v^2 e^{\lambda t}\right] \\ \leq C\left(\int_{\mathcal{D}} \left[u^2 e^{-\lambda t} + v^2 e^{\lambda t} + \theta \left|\nabla_x v\right|^2 e^{\lambda t}\right]\right).\end{aligned} \tag{5.8}$$

The only new point with respect to the proof of uniqueness is the treatment of the boundary term. We now compute the time–boundary contribution coming from

$$A := \frac{1}{2}\int_{\mathcal{D}} \partial_t\Big(u^2 e^{-\lambda t} - a_{ij}(\partial_i v)(\partial_j v)e^{\lambda t} - v^2 e^{\lambda t}\Big)\,\mathrm{d}t\,\mathrm{d}x.$$

By Fubini's theorem $\mathcal{D} = \{(t,x) : x \in V,\ 0 < t < \tau(x)\}$, we can write

$$\begin{aligned}2A &= \int_V \int_0^{\tau(x)} \partial_t\Big(u^2 e^{-\lambda t} - a_{ij}(\partial_i v)(\partial_j v)e^{\lambda t} - v^2 e^{\lambda t}\Big)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_V \left[u^2 e^{-\lambda t} - a_{ij}(\partial_i v)(\partial_j v)e^{\lambda t} - v^2 e^{\lambda t}\right]_{t=0}^{t=\tau(x)}\,\mathrm{d}x \\ &= \int_V \Big(u^2(\tau(x),x)e^{-\lambda\tau(x)} - a_{ij}(\partial_i v)(\partial_j v)(\tau(x),x)e^{\lambda\tau(x)} - v^2(\tau(x),x)e^{\lambda\tau(x)} \\ &\qquad - u^2(0,x) + a_{ij}(\partial_i v)(\partial_j v)(0,x) + v^2(0,x)\Big)\,\mathrm{d}x.\end{aligned}$$

By construction of v we have $v(0,x) = 0$ and $v(\tau(x),x) = 0$ for all $x \in V$, hence the terms involving v^2 vanish at both $t = 0$ and $t = \tau(x)$, and do not contribute to the boundary integral. What remains is a contribution involving $u(\tau(x),x)$ and $\nabla_x v(\tau(x),x)$.

On the hypersurface $S = \{(t,x) : t = \tau(x)\}$ we have $v(\tau(x),x) = 0$ for all $x \in V$. Differentiating this identity with respect to x_i and using the chain rule gives

$$0 = \partial_i\big(v(\tau(x),x)\big) = (\partial_i\tau)(x)\,(\partial_t v)(\tau(x),x) + (\partial_i v)(\tau(x),x).$$

Hence,

$$(\partial_i v)(\tau(x),x) = -(\partial_i\tau)(x)\,(\partial_t v)(\tau(x),x) = (\partial_i\tau)(x)\,u(\tau(x),x)\,e^{-\lambda\tau(x)}.$$

Substituting this expression for $\partial_i v(\tau(x),x)$ into the boundary term yields

$$\frac{1}{2}\int_V u^2(\tau(x),x)\,e^{-\lambda\tau(x)}\Big[1 - a_{ij}(x)(\partial_i\tau)(x)(\partial_j\tau)(x)\Big]\,\mathrm{d}x.$$

By the spacelike condition

$$a_{ij}(x)(\partial_i\tau)(x)(\partial_j\tau)(x) < 1 \quad \text{for all } x \in V,$$

the bracket is nonnegative, and therefore the whole boundary term is nonnegative. Combining this in (5.8) and choosing $\lambda > 0$ large enough to absorb the right-hand side, we obtain

$$\int_{\mathcal{D}} u^2 e^{-\lambda t} = 0,$$

which implies $u = 0$ in $\mathcal{D}$ and completes the proof. □

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Remark 5.14. 1. We say that the equation has *propagation speed at most* $c \geq 0$ if, for every $x_0 \in \mathbb{R}^n$, $R > 0$ and every solution u with

$$\operatorname{supp}\psi_0,\ \operatorname{supp}\psi_1 \subset B(x_0, R),$$

one has

$$\operatorname{supp} u(t,\cdot) \subset B(x_0, R + c|t|) \quad \text{for all } t \in \mathbb{R}.$$

The *maximal propagation speed* is the infimum of all such c.
Assume now that the principal coefficients of L satisfy

$$a_{ij}(x)\,\xi_i\xi_j \leq \nu|\xi|^2 \quad \text{for all } x \in U,\ \xi \in \mathbb{R}^n,$$

for some constant $\nu > 0$. Then the maximal propagation speed is at most $\sqrt{\nu}$.
Indeed, fix $x_0 \in \mathbb{R}^n$ and $R > 0$, and assume $\operatorname{supp}\psi_0, \operatorname{supp}\psi_1 \subset B(x_0, R)$. Let $c > \sqrt{\nu}$ and take $t > 0$ and $x \in \mathbb{R}^n$ with $|x - x_0| > R + ct$. Set

$$d := \operatorname{dist}\big(x, B(x_0, R)\big) = |x - x_0| - R > ct.$$

Choose Λ with

$$\frac{1}{c} < \Lambda < \nu^{-1/2}.$$

Then $t/\Lambda < ct < d$, so we can pick r such that $t/\Lambda < r < d$ and set $V := B(x, r)$. Thus $V \cap B(x_0, R) = \emptyset$, so the initial data (assume source $f \equiv 0$ for simplicity) vanish on V. Choose $0 < \varepsilon < r$ so that

$$\frac{t}{r - \varepsilon} < \Lambda,$$

and then pick $t_1 > t$ so close to t that

$$\frac{t_1}{r - \varepsilon} < \Lambda$$

still holds. On V first consider the cone

$$\tau_0(y) := \frac{t_1}{r}\big(r - |y - x|\big), \qquad y \in V,$$

which satisfies $\tau_0 > 0$ in V, $\tau_0 = 0$ on ∂V, and $\tau_0(x) = t_1 > t$, but is not smooth at $y = x$. We can make it smooth by choosing $\theta \in C^\infty([0, r])$ such that

$$\theta(0) = 0, \quad \theta(r) = r, \quad \theta(s) = s \text{ for } s \geq \varepsilon, \quad 0 \leq \theta'(s) \leq \frac{r}{r - \varepsilon} \text{ for all } s \in [0, r].$$

Define $\tau \in C^\infty(\overline{V})$ by

$$\tau(y) := t_1\Big(1 - \frac{\theta(|y - x|)}{r}\Big), \qquad y \in V.$$

Then $\tau > 0$ in V, $\tau = 0$ on ∂V, and $\tau(x) = t_1 > t$. Moreover, for $y \neq x$ we have

$$\nabla_y\tau(y) = -\frac{t_1}{r}\,\theta'(|y - x|)\,\frac{y - x}{|y - x|}, \qquad |\nabla_y\tau(y)| = \frac{t_1}{r}\,\theta'(|y - x|) \leq \frac{t_1}{r - \varepsilon} < \Lambda.$$

Hence, for all $y \in V$,

$$a_{ij}(y)\,\partial_i\tau(y)\,\partial_j\tau(y) \le \nu|\nabla\tau(y)|^2 \le \nu\Lambda^2 < 1.$$

Thus the hypersurface

$$S = \{(s, y) : s = \tau(y)\}$$

is spacelike and

$$\mathcal{D}_\tau := \{(s, y) : y \in V,\ 0 < s < \tau(y)\}$$

is a domain of dependence in the sense of Theorem 5.13. Since the data vanish on V, Theorem 5.13 implies $u = 0$ in $\mathcal{D}_\tau$. Moreover, $x \in V$ and $0 < t < \tau(x) = t_1$, so $(t, x) \in \mathcal{D}_\tau$, and hence $u(t, x) = 0$.
Therefore $u(t, x) = 0$ whenever $t > 0$ and $|x - x_0| > R + ct$. As $c > \sqrt{\nu}$ was arbitrary, we conclude that

$$\operatorname{supp} u(t, \cdot) \subset B\big(x_0, R + \sqrt{\nu}\, t\big) \quad \text{for all } t \ge 0,$$

i.e. the maximal propagation speed is at most $\sqrt{\nu}$.

5.3 First-order hyperbolic PDEs and the method of characteristics

The previous discussion about finite speed of propagation shows that there is a local “causal cone” for the operator: disturbances at a point (t_0, x_0) cannot influence points outside a cone whose boundary is given (in the principal part) by the vanishing of the symbol

$$p(x, \tau, \xi) = -\tau^2 + a_{ij}(x)\xi_i\xi_j.$$

More precisely, the spacelike condition used in Theorem 5.13 reads $p(x, 1, -\nabla\tau) < 0$, and its limiting case $p = 0$ describes the characteristic directions. After normalising $\tau = 1$, this corresponds to directions ξ satisfying $a_{ij}(t, x)\,\xi_i\xi_j = 1$, which generate the characteristic cones.

In the simplest case of the constant-coefficient wave equation, we proved that a perturbation at $(0, x_0)$ in the data can only influence points (t, x) with $|x - x_0| \le |t|$. Thus the boundary of the region where influence may occur is $\{(t, x) : t > 0,\ |x - x_0| = t\}$, the “light cone” with vertex at $(0, x_0)$, generated by straight rays $x(t) = x_0 + t\sigma$ with $\sigma \in \mathbb{S}^{n-1}$.

This can be related to a formal factorisation of the wave operator

$$\Box = \partial_{tt} - \Delta_x = \big(\partial_t + i\sqrt{-\Delta_x}\big)\big(\partial_t - i\sqrt{-\Delta_x}\big),$$

where each factor corresponds formally to propagation along one family of characteristics. The operators $\sqrt{-\Delta_x}$ are non-local pseudo-differential operators, and we do not study them here. However, in dimension $n = 1$ the situation is more explicit: information propagates along the lines $x_0 \pm t$, which are called the **bicharacteristics** of the PDE. They correspond to the local factorisation $\partial_{tt} - \partial_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)$ and to the rewriting of the PDE as a first-order (system of) **transport equations** $\partial_t v + \partial_x v = 0$ for $v := \partial_t u - \partial_x u$. In this section, we focus on the simpler case of a single **scalar transport equation**, for which only one characteristic trajectory emanates from each point, and we describe the **method of characteristics** for constructing solutions, strong and weak. This is the simplest way to see why nonlinear hyperbolic PDEs can develop singularities.

5.3.1 Some examples

Consider for $c \in \mathbb{R}$ the **scalar linear** 1**d transport equation with constant coefficient**:

$$\begin{cases} \partial_t u + c\,\partial_x u = 0, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.9}$$

It models transport of particles (or molecules, or cars, or thingies…) on a line with algebraic velocity c, and u is the density (or velocity, or else…) of such thingies at time t and point x along the line. It is easy to check that given $u_0 \in C^1(\mathbb{R})$ there is a unique classical solution $u \in C^1(\mathbb{R}^2)$ given by $u(t,x) = u_0(x - ct)$. One can also solve the equation with a source term f by the Duhamel principle. Also, as we shall see, given $u_0 \in L^\infty(\mathbb{R})$, there is a unique weak solution $u \in L^\infty(\mathbb{R}^2)$ given again by $u(t,x) = u_0(x - ct)$ (we will do it in the more general case of variable coefficients).

When $c = c(x)$, we obtain a transport equation with **variable transport velocity**. When $c = c[u]$ depends on the solution, we obtain a **nonlinear transport equation**. The **Burgers equation** (1948) is the Euler equation in 1D for the velocity field u of the fluid when the density is constant (the nonlinearity is caused by the convection term):

$$\partial_t u + u\partial_x u = \partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0. \tag{5.10}$$

The **Burgers equation** (1948) is the one-dimensional Euler equation for the velocity field u of a fluid with constant density; its nonlinearity comes from the convection term:

$$\partial_t u + u\,\partial_x u \;=\; \partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0. \tag{5.11}$$

More generally, equations of the form $\partial_t u + \partial_x\big(C(u)\,u\big) = 0$ with non-constant $C = C(u(t,x))$ are a standard toy model for nonlinear transport. As an example, consider a one-lane road with no entry or exit. Assume the typical car length is much smaller than the observation scale, so that traffic can be modeled as a “continuum”. Let $u(t,x)$ be the car density, $v(t,x)$ the velocity, and $q(t,x)$ the flux at (t,x) (the rate at which cars pass to point x at time t, in average). The number of cars in $[x, x+\delta x]$ at time t is $u(t,x)\,\delta x$, and at time $t+\delta t$ it is $u(t+\delta t, x)\,\delta x$. By conservation of cars, this change equals the inflow at x minus the outflow at $x + \delta x$ between times t and $t + \delta t$:

$$\frac{u(t+\delta t, x) - u(t,x)}{\delta t} + \frac{q(t, x+\delta x) - q(t,x)}{\delta x} = 0.$$

Letting $\delta t, \delta x \to 0$ yields $\partial_t u + \partial_x q = 0$. This **continuity equation** expresses conservation of the number of cars and is a prototype of **conservation laws**, a subclass of hyperbolic equations.

To close the model, we need a **state equation** relating q to u. Here it is given by driver behavior: we write $q(t,x) = u(t,x)\,C(t,x)$ (number of cars crossing x per unit time is density of cars times speed C), and assume $C = C(u(t,x))$, i.e. the mean speed depends only on the locally observed density. The function $C(r)$ is determined experimentally; typically it decreases, as the speed is higher when there is little traffic and decreases as the density of cars increases. Even such a simple PDE exhibits formation of discontinuities, which correspond to the formation of traffic jams.

5.3.2 The linear scalar transport equation with variable coefficients

Given $u_0 \in C^1(\mathbb{R}^n)$, we consider the following equation

$$\begin{cases} \partial_t u(t,x) + F(t,x) \cdot \nabla_x u(t,x) = 0, & t \in \mathbb{R},\ x \in \mathbb{R}^n \\ u(t=0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{5.12}$$

for a variable propagation speed given by a vector field $F(t,x) \in \mathbb{R}^n$. We assume $F \in C^1(\mathbb{R} \times \mathbb{R}^d)$, and $|\nabla_x F(t,x)| \le L$ for all $t, x \in \mathbb{R}^2$, for some constant $L > 0$ (the assumption on the gradient could be replaced in the sequel by $F(t,x) \le L\,(1+|x|)$ for all $t, x \in \mathbb{R}^2$).

Definition 5.15. For each $s \in \mathbb{R}$ and $x \in \mathbb{R}^n$, the **characteristic** of the transport equation (5.12) passing through (s,x) is the unique solution

$$y : t \mapsto y(t) \in \mathbb{R}^n$$

of the ODE

$$\dot{y}(t) = F(t, y(t)), \qquad y(s) = x.$$

We denote this solution by $y(t) = Z_{s,t}(x)$, and call the family $(Z_{s,t})_{s,t\in\mathbb{R}}$ the flow of characteristics.

We know from the ODE theory, since we are assuming in particular F continuous in t and globally Lipschitz in x, these trajectories exist for all $s, t \ge 0$ and all starting point $x \in \mathbb{R}^n$. For each $s, t \in \mathbb{R}$ the map

$$Z_{s,t} : \mathbb{R}^n \to \mathbb{R}^n, \qquad x \mapsto Z_{s,t}(x)$$

is a C^1–bijection with inverse $Z_{t,s}$, hence a C^1–diffeomorphism. In particular, trajectories cannot cross: if $Z_{s,t}(x_1) = Z_{s,t}(x_2)$, uniqueness backward in time implies $x_1 = x_2$. Although there is in general no semigroup structure (i.e. one cannot write $Z_{s,t} = S_{t-s}$ for a one-parameter family $(S_\tau)_{\tau\in\mathbb{R}}$ with $S_0 = \mathrm{Id}$ and $S_{\tau_2} \circ S_{\tau_1} = S_{\tau_1+\tau_2}$, since F depends on t), the flow still satisfies

$$Z_{t_1,t_2} \circ Z_{t_0,t_1} = Z_{t_0,t_2} \quad \text{for all } t_0, t_1, t_2 \in \mathbb{R},$$

again by uniqueness of solutions.

Theorem 5.16. *Given $u_0 \in C^1(\mathbb{R}^n)$ and $F \in C^1(\mathbb{R}^{n+1})$ with bounded spatial gradient, the Cauchy problem (5.12) admits a unique global classical solution $u \in C^1(\mathbb{R}^{n+1})$. Denoting by $(Z_{s,t})_{s,t\in\mathbb{R}}$ the flow generated by F, the solution is given in **implicit form** by*

$$u\big(t, Z_{0,t}(x)\big) = u_0(x),$$

*and equivalently in **explicit form** by*

$$u(t,x) = u_0\big(Z_{t,0}(x)\big), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

Remark 5.17. The basic idea behind the theorem is that the solution is transported along the characteristic trajectories, and hence remains constant on each curve $t \mapsto Z_{0,t}(x)$. This method of characteristics provides a concrete bridge between ODE and PDE theory, and shows in some cases how to view a PDE as a continuum of ODEs.

More generally, for the inhomogeneous equation

$$\partial_t u(t,x) + F(t,x) \cdot \nabla_x u(t,x) = f(t,x),$$

with $f \in C^0(\mathbb{R}^{n+1})$, one has along characteristics $\frac{d}{dt} u\big(t, Z_{0,t}(x)\big) = f\big(t, Z_{0,t}(x)\big)$, so that by Duhamel's principle

$$u\big(t, Z_{0,t}(x)\big) = u_0(x) + \int_0^t f\big(\tau, Z_{0,\tau}(x)\big)\, d\tau,$$

or equivalently

$$u(t,x) = u_0\big(Z_{t,0}(x)\big) + \int_0^t f\big(\tau, Z_{t,\tau}(x)\big)\, d\tau.$$

Proof of Theorem 5.16. By uniqueness, any $u \in C^1$ solution to (5.12) satisfies by chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[u\left(t, Z_{0,t}(x)\right)\right] &= (\partial_t u)\left(t, Z_{0,t}(x)\right) + (\nabla_x u)\left(t, Z_{0,t}(x)\right) \cdot \partial_t Z_{0,t}(x) \\ &= (\partial_t u + F \cdot \nabla_x u)\left(t, Z_{0,t}(x)\right) = 0,\end{aligned}$$

which shows $u(t, Z_{0,t}(x)) = u(0, Z_{0,0}(x)) = u_0(x)$ since $Z_{0,0}(x) = x$ and determines the solution since $Z_{0,t}$ is a C^1-diffeomorphism for any $t \in \mathbb{R}$. To prove existence, consider $w(t,x) := u_0\left(Z_{t,0}(x)\right)$ for $t, x \in \mathbb{R}$, which is C^1 in both variables by composition, and satisfies the initial condition since $Z_{0,0} = \mathrm{Id}$. It also satisfies $w(t, Z_{0,t}(x)) = u_0\left(Z_{t,0} \circ Z_{0,t}(x)\right) = u_0(x)$. By differentiating in time this last equation one gets by chain rule $(\partial_t w + F \cdot \nabla w)(t, Z_{0,t}(x)) = 0$ for all $t, x \in \mathbb{R}$ and every $y \in \mathbb{R}^n$ can be written as $y = Z_{0,t}(x)$ since $Z_{0,t} : \mathbb{R}^n \to \mathbb{R}^n$ is bijective for any $t \in \mathbb{R}$, thus we conclude that $\partial_t w + F \cdot \nabla w = 0$ everywhere, so $u = w$ is a solution. □

To introduce a notion of **weak solution**, we restrict to the case of F in **divergence-free** vector fields F, this makes the equation in a form of conservation laws. If u and F are smooth, then $\partial_t u + F \cdot \nabla_x u = 0$ is equivalent to $\partial_t u + \nabla_x \cdot (Fu) = 0$ whenever $\nabla_x \cdot F \equiv 0$. This motivates restricting to **divergence-free** vector fields F and taking the weak formulation of the conservative equation.

Definition 5.18. Let $u_0 \in L^\infty(\mathbb{R}^n)$ and $F \in C^1(\mathbb{R}^{n+1})$ with $\sup_{(t,x)\in\mathbb{R}^{n+1}} |\nabla_x F(t,x)| < \infty$ and zero divergence $\nabla_x \cdot F \equiv 0$. A weak L^∞ solution to (5.12) is a function $u \in L^\infty(\mathbb{R} \times \mathbb{R}^n)$ such that for all $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}^n)$, (we use $\mathbb{R}_+ = [0, \infty)$, so the test function can be non-zero at the $(0, x)$)

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^n} u(t,x) \left[\partial_t \varphi(t,x) + F(t,x) \cdot \nabla_x \varphi(t,x)\right] dt\, dx + \int_{\mathbb{R}^n} u_0(x)\, \varphi(0,x)\, dx = 0. \quad (5.13)$$

Theorem 5.19 (Weak-strong uniqueness principle)**.** *Let $u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $F \in C^1(\mathbb{R}^{n+1})$ with the assumptions above. Any classical solution to* (5.12) *is also a weak solution, and any weak solution that is C^1 is also a classical solution.*

Proof of Theorem 5.19. A classical solution $u(t,x) = u_0(Z_{t,0}(x)) \in L^\infty(\mathbb{R}^{n+1})$ by construction, and given $\varphi \in C^1_c(\mathbb{R}^{n+1})$ we recover the weak formulation by integrating the PDE against φ and integrating by parts (check it). If $u \in C^1(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1})$ is a weak solution, we first consider (5.13) with test function $\varphi \in C^1_c(\mathbb{R}^* \times \mathbb{R}^n)$ (who support avoids the initial time) and compute by integration by parts (note all integration by parts use $\nabla_x \cdot F = 0$)

$$0 = \int_{\mathbb{R}^{n+1}} u\big(\partial_t \varphi + F \cdot \nabla_x \varphi\big)\, \mathrm{d}t\, \mathrm{d}x = -\int_{\mathbb{R}^{n+1}} \big(\partial_t u + F \cdot \nabla_x u\big) \varphi\, \mathrm{d}t\, \mathrm{d}x$$

which implies that the PDE holds pointwise. Second we consider $\varphi \in C_c^1(\mathbb{R}^{n+1})$ and compute $\int_{\mathbb{R}^n} (u_0(x) - u(0,x))\, \varphi(0,x)\, \mathrm{d}x = 0$. Since it is true for any $\varphi(0,x) \in C^1(\mathbb{R}^n)$ we deduce that the initial condition is satisfied. □

Theorem 5.20. *Let $u_0 \in L^\infty(\mathbb{R}^n)$ and $F \in C^1(\mathbb{R}^{n+1})$ with the assumptions as in the definition of weak solution. There is a unique global weak solution $u \in L^\infty(\mathbb{R} \times \mathbb{R}^n)$ to (5.12), given again by $u(t,x) = u_0\,(Z_{t,0}(x))$.*

Proof of Theorem 5.20. To prove existence, we define $u(t,x) := u_0(Z_{t,0}(x)) \in L^\infty(\mathbb{R}^{n+1})$, then given $\varphi \in C_c^1(\mathbb{R}^{n+1})$ we compute

$$\int_{\mathbb{R}^{n+1}} u(t,x)\big(\partial_t\varphi + F\cdot\nabla_x\varphi\big)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}^{n+1}} u_0(Z_{t,0}(x))\Big[\partial_t\varphi + F\cdot\nabla_x\varphi\Big]\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^{n+1}} u_0(X)\Big[\partial_t\varphi(t,Z_{0,t}(X)) + F(t,Z_{0,t}(X))\cdot\partial_x\varphi(t,Z_{0,t}(X))\Big]\,\mathrm{d}t\,\mathrm{d}X$$
$$= \int_{\mathbb{R}^{n+1}} u_0(X)\frac{\mathrm{d}}{\mathrm{d}t}\Big[\varphi(t,Z_{0,t}(X))\Big]\,\mathrm{d}t\,\mathrm{d}X = -\int_{\mathbb{R}^n} u_0(X)\,\varphi(0,X)\,\mathrm{d}X$$

where we have used the change of variable $X = Z_{t,0}(x)$, the chain rule $\frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi(t,Z_{0,t}(X))\big] = \partial_t\varphi(t,Z_{0,t}(X)) + F\cdot\nabla_x\varphi(t,Z_{0,t}(X))$, and an integration by parts in t (keeping X fixed). This proves that u satisfies the weak formulation. To prove uniqueness, it is enough by linearity to prove that the weak solution u with zero initial data $u_0 = 0$ must be the zero solution. We follow the **dual method**. To prove that $u = 0$, it is enough to prove that for any $\psi \in C_c^1(\mathbb{R}^{n+1})$, there exists $\varphi \in C_c^1(\mathbb{R}^{n+1})$ so that $\partial_t\varphi + F\cdot\nabla_x\varphi = \psi$ for $t \geq 0$, then the weak formulation on u (with zero initial data) implies $\int_{\mathbb{R}_+\times\mathbb{R}^n} u\psi = 0$, and $u = 0$ on $t \geq 0$ (and a symmetric argument proves $u = 0$ on $t \leq 0$). To prove the claim, we use the proof of existence of classical solutions with a source term and define, for $t \geq 0$ and given $\psi \in C_c^1(\mathbb{R}^{n+1})$,

$$\varphi(t,x) := \varphi_0(Z_{t,0}(x)) + \int_0^t \psi(s,Z_{t,s}(x))\,\mathrm{d}s = \int_T^t \psi(s,Z_{t,s}(x))\,\mathrm{d}s$$

with the choice of initial data $\varphi_0(x) := -\int_0^T \psi(s,Z_{0,s}(x))\,\mathrm{d}s$. Since ψ is compactly supported, there exist $T > 0$ and $R > 0$ such that $\operatorname{supp}\psi \subset [0,T]\times B_R(0)$. Moreover, since F is bounded on $[0,T]\times\mathbb{R}^n$, say $|F(t,x)| \leq M$, the flow $(Z_{0,s})_{s\in[0,T]}$ satisfies

$$|Z_{0,s}(x) - x| = \left|\int_0^s F\big(\tau, Z_{0,\tau}(x)\big)\,d\tau\right| \leq Ms \leq MT.$$

If we choose $R' := R + MT$, then $|x| \geq R'$ implies $|Z_{0,s}(x)| \geq |x| - MT \geq R$ for all $s \in [0,T]$, hence $Z_{0,s}(x) \notin B_R(0)$. In particular, $\psi\big(s, Z_{0,s}(x)\big) = 0$ for all $s \in [0,T]$ and $|x| \geq R'$, and thus

$$\varphi_0(x) = -\int_0^T \psi\big(s, Z_{0,s}(x)\big)\,ds = 0 \quad \text{for } |x| \geq R',$$

so $\varphi_0 \in C_c^1(\mathbb{R}^n)$. Arguing as above with the uniform Lipschitz bounds for $(Z_{t,s})$ on $[0,T]^2$ and the spatial support of ψ, we also find $R'' > 0$ such that $\varphi(t,x) = 0$ whenever $|x| \geq R''$. Hence $\varphi \in C_c^1(\mathbb{R}_+\times\mathbb{R}^n)$, as required. □

5.3.3 Nonlinear scalar monodimensional transport equations

Let us consider the nonlinear conservation law

$$\partial_t u + \partial_x \big(\mathfrak{f}(u)\big) = \partial_t u + \mathfrak{f}'(u)\,\partial_x u = 0, \tag{5.14}$$

where $\mathfrak{f}$ is the **flux** and $F(t,x) := \mathfrak{f}'(u(t,x))$ is the **speed of propagation**. Given $\mathfrak{f} \in C^2(\mathbb{R})$ with $\mathfrak{f}' \in L^\infty(\mathbb{R})$ and $u_0 \in C^1(\mathbb{R})$, we can define the characteristic trajectories $Z_{s,t}(x)$ and the notion of classical solution as above. Note however that the characteristics now solve

$$\partial_t Z_{s,t}(x) = F\big(t, Z_{s,t}(x)\big) = \mathfrak{f}'\big(u(t, Z_{s,t}(x))\big), \qquad Z_{s,s}(x) = x,$$

and therefore depend on the (unknown) solution u. This *nonlinear loop* means that the characteristics cannot be computed independently of u, and so cannot be used a priori to construct the solution as in the linear case.

The key **a priori estimate** is that, if u is a classical solution, then u stays **constant along characteristic trajectories**:

$$\begin{aligned}\frac{d}{dt}\big[u(t, Z_{0,t}(x))\big] &= \partial_t u\big(t, Z_{0,t}(x)\big) + \partial_x u\big(t, Z_{0,t}(x)\big)\,\partial_t Z_{0,t}(x)\\ &= \big(\partial_t u + \mathfrak{f}'(u)\,\partial_x u\big)\big(t, Z_{0,t}(x)\big) = 0,\end{aligned}$$

which implies

$$u\big(t, Z_{0,t}(x)\big) = u\big(0, Z_{0,0}(x)\big) = u_0(x),$$

so the value of u along each characteristic is completely determined by the initial data. Hence the characteristic equation becomes

$$\partial_t Z_{0,t}(x) = \mathfrak{f}'\big(u(t, Z_{0,t}(x))\big) = \mathfrak{f}'\big(u_0(x)\big),$$

and the characteristics are in fact straight lines:

$$Z_{0,t}(x) = x + t\,\mathfrak{f}'\big(u_0(x)\big).$$

When $\mathfrak{f}' \circ u_0$ is decreasing, one checks (see Exercise 4.8) that there is a time

$$T_* = -\Big[\min_{x\in\mathbb{R}} \big(\mathfrak{f}' \circ u_0\big)'(x)\Big]^{-1}$$

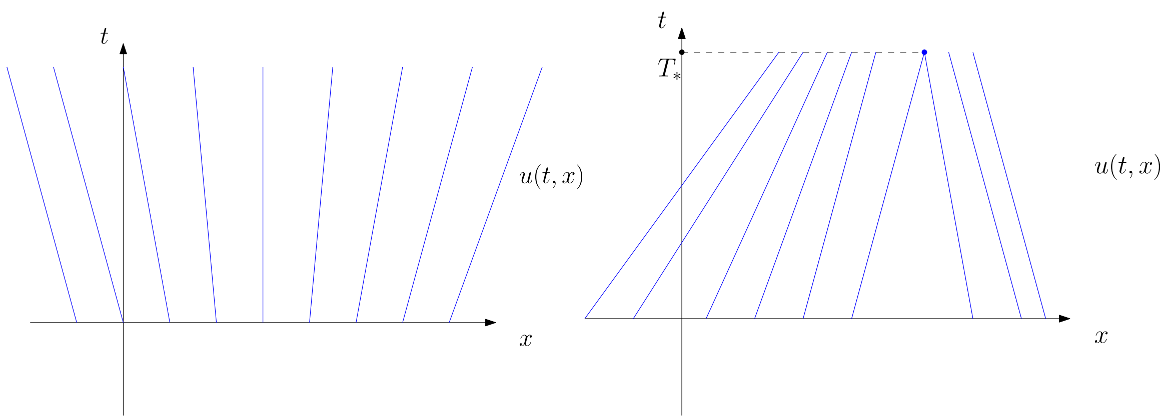

Figure 1: Picture of characteristics in the cases $f' \circ u_0$ increasing and then decreasing.

at which this classical solution breaks down: if the characteristic formula above were to remain valid beyond T_*, two distinct initial points would be mapped to the same position, forcing u to take two different values there. This corresponds to the formation of a **shock** (or **caustic**), and the solution becomes discontinuous. To continue the solution beyond T_* in a unique way, the notion of weak L^∞ solution is not sufficient (weak solutions are not unique); one introduces instead **entropy solutions** in the sense of Kruzhkov (1970) (see Exercise 4.11).

A Appendix on Hilbert spaces

Everything needed in the course on the theory of Banach and Hilbert spaces can be found in the appendix D of the book by Evans. We give the proof of the Fredholm alternative for compact operators.

Proof of Theorem 4.14 (Fredholm alternative for compact operators) (Not examinable). (i) With $T := \mathrm{Id} - K$, the unit ball of $\operatorname{Ker} T$ is compact since $u = Ku$ for $u \in \operatorname{Ker} T$ and K is compact; hence $\operatorname{Ker} T$ is finite-dimensional.

(ii) Let (u_m) converge in $\operatorname{Im} T$, so $u_m = v_m - Kv_m$ for some (v_m). Decompose $v_m = w_m + h_m$ with $w_m \in \operatorname{Ker} T$ and $h_m \in (\operatorname{Ker} T)^\perp$. Since $Tw_m = 0$, we have $u_m = h_m - Kh_m$. If (h_m) is bounded, compactness of K gives a subsequence (Kh_{m_k}) converging, hence (h_{m_k}) converges and $u_\infty \in \operatorname{Im} T$. If (h_m) is unbounded, set $\tilde{h}_m = h_m/\|h_m\|$; then $\tilde{h}_m - K\tilde{h}_m \to 0$, so a subsequence converges to some $h_\infty \in \operatorname{Ker} T$ with $\|h_\infty\| = 1$, contradicting $\tilde{h}_m \in (\operatorname{Ker} T)^\perp$. Thus $\operatorname{Im} T$ is closed.

(iii) Using (ii) and $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$, we get $\operatorname{Im} T = (\operatorname{Ker} T^*)^\perp$. The adjoint of a compact operator is compact.

(iv) By (iii) it suffices to prove: if $\operatorname{Im} T = H$, then $\operatorname{Ker} T = \{0\}$. If $\operatorname{Ker} T^k \subsetneq \operatorname{Ker} T^{k+1}$ for all k, one builds an orthonormal sequence (u_k) with $u_k \in \operatorname{Ker} T^{k+1} \cap (\operatorname{Ker} T^k)^\perp$, and then (Ku_k) has no convergent subsequence, contradicting compactness. Hence $\operatorname{Ker} T^{k_0} = \operatorname{Ker} T^{k_0+1}$ for some k_0; if $k_0 \geq 1$, pick $0 \neq u \in \operatorname{Ker} T^{k_0} \setminus \operatorname{Ker} T^{k_0-1}$, write $u = Tv$ (surjectivity), and deduce $v \in \operatorname{Ker} T^{k_0}$, a contradiction. Thus $\operatorname{Ker} T = \{0\}$; the converse follows by applying the same argument to K^*.

(v) Both kernels are finite-dimensional by (i). If one is trivial, (iii)–(iv) give that the other is trivial. Otherwise, reduce dimensions one by one by adding a rank-one perturbation and use induction. □

The following is a list of results in the theory of Hilbert spaces that we used in the course.

- *Riesz representation theorem.* For any Hilbert space H and $F \in H^*$ there exists a unique $y \in H$ with $F(v) = (v, y)$ for all $v \in H$, and $\|F\| = \|y\|$. Hence $H^* \cong H$.
- *Adjoint of a bounded operator.* For bounded $T : H \to H$ there is a unique bounded T^* such that $(Tu, v) = (u, T^*v)$ for all $u, v \in H$. In particular $(T^*)^* = T$, $\|T^*\| = \|T\|$, $(TS)^* = S^*T^*$, and $(\operatorname{Im} T)^\perp = \operatorname{Ker} T^*$ so $\overline{\operatorname{Im} T} = (\operatorname{Ker} T^*)^\perp$.
- *Spectrum of compact operators.* If K is compact on infinite dimensional H, every nonzero spectral value is an eigenvalue of finite multiplicity and 0 is the only possible accumulation point; if K is selfadjoint, all eigenvalues are real (and there is an orthonormal basis of eigenvectors).

Also, in the theorem of existence of weak solutions for wave type equations we used the following basic spectral property of compact self-adjoint operators.

Lemma A.1. *Any non-zero compact self-adjoint operator K on a Hilbert space has at least one non-zero eigenvalue. This contradicts the fact that by construction K has no eigenvalues on E. This contradiction shows that $V = H^1_0(U)$, i.e. (φ_k) is dense in $H^1_0(U)$.*

Proof. (Not examinable) First, one shows

$$\|K\| = \sup_{\|u\|=1} (Ku, u),$$

which follows from the polarization identity. Choose a sequence (u_m) with $\|u_m\| = 1$ such that $(Ku_m, u_m) \to \|K\|$. Since K is compact, (Ku_m) has a convergent subsequence (still denoted (Ku_m)), say $Ku_m \to v$ in the Hilbert space. Passing to the limit in

$$(Ku_m, u_m) \to (v, u_*), \qquad \text{for some weak limit } u_* \text{ of } (u_m),$$

one checks that $v = \|K\| \, u_*$ (or $v = -\|K\| \, u_*$), hence $u_* \neq 0$ and $Ku_* = \pm\|K\| \, u_*$. Thus $\pm\|K\|$ is an eigenvalue of K, □